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ГОСУДАРСТВЕННОЕ ИЗДАТЕЛЬСТВО
ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ

Москва

of

ELASTICITY

by

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Introduction

The theory of elasticity developed into an important branch of mathematical physics in the first half of the 19th century. Before that the scientists of the 17th and 18th centuries—Galileo Galilei, E. Mariotte, R. Hooke, J. Bernoulli, L. Euler, C. A. Coulomb and others—had worked out, in some detail, the theory of the bending of thin elastic bars. At the beginning of the 19th century J. L. Lagrange and Sophie Germain solved the problem of the bending and vibrations of thin elastic plates. Some characteristics of such thin elastic bodies made it possible to simplify considerably the formulation and solution of problems concerning deformation under the action of external forces without going too deep into the essence of the phenomena taking place in a material. The beginning of the 19th century was marked by vast progress in mathematical analysis, partly due to a variety of new important problems arising in physics and requiring the application of the complex mathematical apparatus and its further development. This led to the emergence of a special trend in physics, called mathematical physics. Among the great number of problems confronting this new branch of science it is necessary to mention the need for a profound investigation of the properties of elastic materials and for the construction of a mathematical theory which would permit studying as completely as possible the internal forces occurring in an elastic body under the action of external forces, as well as the deformation of a body, i.e., the change of its shape. Investigations of this kind were also urgently needed to meet the demands of rapid technological progress entailed by railway building and engineering. These demands ensued from the necessity of devising the theoretical methods for designing strong structural members and machines. In 1825 the prominent French engineer and scientist C.L.M.H. Navier published *Résumé des leçons sur l'application de la mécanique. P. I. Contenant les leçons sur la résistance des matériaux* (Paris, F. Didot), based on the experimental data available at that time and the approximate

theories indicated above. In Russia an analogous treatise by N. F. Jastrzębski appeared in 1837.

The fundamentals of the theory of elasticity were worked out by the French mathematicians and mechanicians A. L. Cauchy, C.L.M.H. Navier, and S.D. Poisson, who derived the basic differential equations of this theory. One must mention the great difficulty encountered in its development. Mathematical analysis in those days was built on the concept of continuous geometrical space in which it was possible to consider infinitesimal segments and to introduce the processes of differentiation and integration on this basis. The universally recognised Newtonian molecular theory of structure of bodies, on the other hand, represented them as discrete media composed of individual particles that are connected with each other by the forces of mutual attraction and repulsion. It proved to be very difficult to justify the applicability to such media of the apparatus of mathematical analysis, which was essentially connected with the concept of continuous functions capable of receiving indefinitely small (i.e., infinitesimal) increments, and with the possibility of passing to the limit in their summation, i.e., in their integration. On account of this, the first works on the mathematical theory of elasticity gave rise to much discussion; their validity was questioned. However, the fact that even an extremely small volume, presumedly isolated from a body, contains a great number of molecules prompted investigators to appeal to the law of large numbers and to apply the method which was subsequently called statistical; this made it possible to bridge the gap between the continuous space of mathematical analysis and the solid body as a discrete medium. It became possible to apply the powerful apparatus of mathematics to the development of the new branch of physics.

The importance of the application of the theory of elasticity in physics and technical mechanics and the tremendous complexity involved in the stated problems from the viewpoint of mathematical analysis attracted the attention of the greatest investigators of the 19th and 20th centuries to this new branch of science. Besides the founders of the theory of elasticity named above, Cauchy, Navier and Poisson, we might mention such outstanding scientists as M. V. Ostrogradsky, G. Lamé (who published the first course of lectures on the theory of elasticity in 1852), B.P.E. Clapeyron, B. de Saint-Venant, G. Green, J. C. Maxwell, Lord Kelvin (Sir W. Thomson), Lord Rayleigh, J. H. Michell, E. Mathieu, F. S. Jasinsky, S. P. Timoshenko, G. V. Kolosoff, N. I. Muskhelishvili and many others.

In treating statics in theoretical mechanics, it is stated that the conditions of equilibrium of a body or a system of bodies do

not involve internal forces since they are pairwise mutually equilibrated on the basis of Newton's third law of equality of action and counteraction. The theory of elasticity sets forth the problem of determining internal forces in a solid body. These forces represent interaction between molecules; they ensure the existence of a solid body as such, its strength. They also act when no external forces are applied to the body; these forces are not in themselves the object of study in the theory of elasticity; under the action of external forces the body deforms, the mutual position of molecules changes and so do the distances between them; the action of the external forces that produce deformation gives rise to additional internal forces. Statics deals with external forces applied to a body; it is necessary therefore first to convert the internal forces in question into external forces. This is achieved by the so-called *method of sections* which consists in the following (Fig. 1).

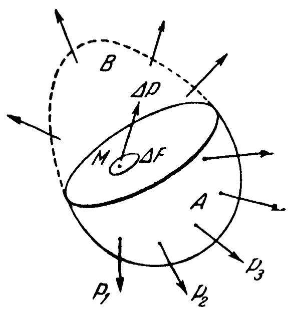

Fig. 1

Let us imagine a body cut into two parts by a plane[1]; this plane intersects the lines of forces of interaction between molecules situated on the two sides of it. If we now imagine that one of the parts of the body has been removed, for example B, the system of forces of interaction applied to molecules of the part A, located near the plane of section, will be unbalanced. The remaining part A, however, is in equilibrium; hence, the system of internal forces brought about by the method of sections and converted now into forces external relative to the part A must be balanced by the system of the remaining external forces P_1, P_2, ... acting on this part of the body.

We now isolate in the plane of section an elementary area ΔF that is very small compared with the dimensions of the section, but very great compared with the distances between the individual molecules of the body; this area is intersected by a large number of lines of action of the discovered internal forces applied to molecules of the part A and exerted by the removed part B. Let ΔP denote their resultant force vector; the ratio

$$\frac{\Delta P}{\Delta F}$$

is called the *average stress* of internal forces in the body across the area ΔF. In order to apply subsequently the apparatus of

[1] A curved surface is often taken instead of a plane.

mathematical analysis, we shall contract the contour of the element ΔF around any of its points, M; then its area ΔF and the resultant force vector will diminish indefinitely; the limit of the ratio

$$\lim_{\Delta F \to 0} \frac{\Delta P}{\Delta F} = p$$

is called *stress of internal forces* or simply the *stress of the point M of the body on the area lying in the plane of the section made.* The notion of stress in the theory of elasticity is a fundamental measure of the intensity of internal forces brought about by the method of sections. The stress is characterised by its magnitude having the dimension |force/length^2|, direction and position, i.e., it has a vector character; it is represented, therefore, as a vector $\mathbf{p}$ applied at the point M to which it corresponds. The vector $\mathbf{p}$ is termed the *total stress* in distinction of the components into which it can be resolved. The basic properties of stresses will be dealt with later, in Chapter I.

The passage, made above, from the average stress across the area to the stress at the point is connected with the imaginary process of reducing the dimensions of the area ΔF to zero, which is necessary for the application of the analysis of infinitesimals. The validity and justification of such a formal process, as already stated above, had long been questioned and discussed by scientists; however, the application of the resulting basic equations of the theory of elasticity to the solution of problems of physics demonstrated fairly soon the efficiency of the methods worked out and brought a series of remarkable results confirmed by experience; this pertains first and foremost to the study of vibrations and propagation of waves (e.g., sound waves) in elastic bodies; some of the simpler problems of this kind are treated in Chapters IV and IX. The middle of the 19th century was especially rich in achievements as regards the development of the theory of elasticity and the solution of problems of importance for physics and technical mechanics; the work of the eminent French investigator Saint-Venant and his pupils played a dominant part here. Under these circumstances, doubts concerning the physical justification of the method of elasticity dealing, as it were, with a continuous solid medium gradually disappeared. From this point of view, it is sometimes said that the theory of elasticity is based on the *hypothesis of the continuous structure of solids.* It must be borne in mind, of course, that this hypothesis is but a *working hypothesis*; it is dictated by the adopted mathematical method of investigation and does not intrude into the branches of physics that are directly concerned with the problems of body structure.

I

Theory of Stress

1. STATE OF STRESS IN A BODY

Beginning the study of the state of stress in a body, let us first introduce a system of notation of stresses which would permit us to distinguish easily between stresses on different areas passing through a given point and be as mnemonic as possible.

There exist several systems of notation; here we shall use the system employed in many courses on the theory of elasticity.

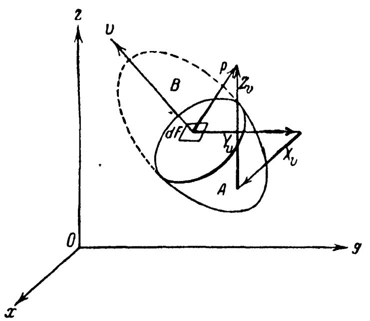

Fig. 2

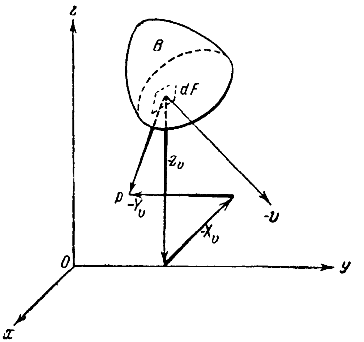

Fig. 3

Imagine (Fig. 2) a solid cut by a plane into two parts A and B. The part B is removed thereafter; we isolate an element of area dF in the plane of the section; the orientation of this area will be characterised by the outward normal to it (outward relative to the part A which has remained).

With the aid of the outward normal we indicate tersely and clearly not only the orientation of the area but also the part of the body (B or A) which is removed after the section has been made and the action of which is subsequently replaced by forces.

Let p be the total stress at a given point on the area dF. Consider any axis x (i.e., a straight line with a specified direc-

tion on it); we shall denote the projection of the stress p on this axis by X_v and read this notation thus: "The projection on the x axis of the stress p on an area with outward normal v". Thus, the main letter X indicates the direction of the axis on which the stress is being projected; the subscript v characterises the orientation of the area on which the stress is being considered.

We select somewhere in space an arbitrary system of rectangular co-ordinates $Oxyz$ with the aid of which we shall determine the position of the points of the body under consideration. Stresses will be determined by their projections on the axes of this system; we shall go on to use it in the analysis of strain.

The projections on the coordinate axes of the total stress p acting on the area dF will be denoted by X_v, Y_v, Z_v. If we remove

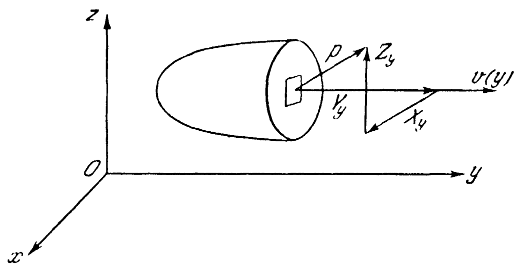

Fig. 4

the part A after the section has been made, the outward normal to the same area dF will be directed opposite; we denote it, therefore, by $-v$. The projections of the total stress on it (Fig. 3) have to be denoted as: X_{-v}, Y_{-v}, Z_{-v}; it is obvious that

$$X_{-v} = -X_v; \quad Y_{-v} = -Y_v; \quad Z_{-v} = -Z_v,$$

since stresses X_v, Y_v, Z_v express the action of the part B on the part A, while stresses X_{-v}, Y_{-v}, Z_{-v} express the action of the part A on the part B; these actions are equal in magnitude, but opposite in sense.

Subsequently we shall frequently make sections perpendicular to one of the co-ordinate axes. Let us make, for instance, a section perpendicular to the axis Oy; we then obtain the following projections of the total stress (Fig. 4):

$$X_y, \; Y_y, \; Z_y.$$

The projection Y_y is normal to the area and is called the *normal stress*; the projections X_y and Z_y, lying in the plane of the area, are called the *shearing stresses*.

Fig. 5*a*, *b* shows the notation of stresses on the areas normal to the other two axes. Bringing these results together, we obtain the

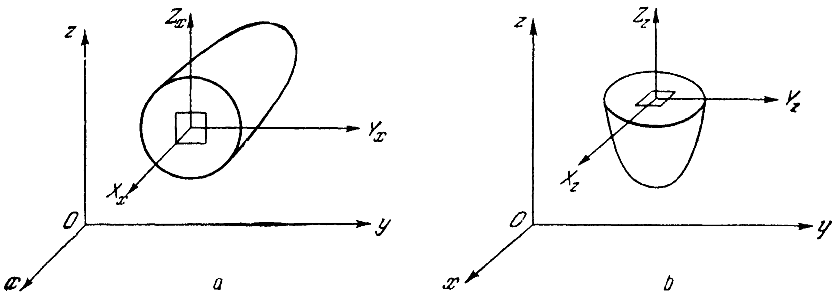

Fig. 5

following system of stresses on the areas parallel to the coordinate planes

$$X_x,\ Y_x,\ Z_x;\quad X_y,\ Y_y,\ Z_y;\quad X_z,\ Y_z,\ Z_z.$$

One can easily see that the stresses

$$X_x,\ Y_y,\ Z_z,$$

in whose notation both letters are the same, are normal stresses. The remaining six stresses are shearing stresses.

In order to gain a better orientation in these new notations the reader is recommended to do the following exercises.

Exercises.

1. Fig. 6 represents the cross section yOA of the wall of a dam which is subjected to water pressure against the face Oy; the face OA does not carry any load.

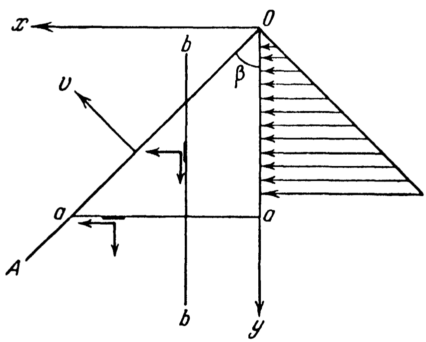

Fig. 6

Designate the normal and shearing stresses on the face Oy and write down what they are equal to (pay attention to the direction of pressure, the direction of the axis Ox and the sign of the projection). Designate also the components of stress on the oblique face OA (the outward normal v) and write down therefrom the conditions that the face OA is free of load. Designate the stresses at the sections *aa* and *bb* when any part of the wall (left or right, upper or lower) is removed after the cut has been made.

2. A cylindrical body (Fig. 7) is twisted by forces applied to the end cross sections. Designate the stresses at any point k of the cross section ln and write down the condition that the normal stress is absent. Designate the stress at any point m of the lateral surface and write down the conditions that this surface is not loaded.

3. A cylindrical rod (Fig. 8) is bent by forces applied to the end cross section pq. Write down the conditions that the lateral surface is free of load (see the preceding exercise), designate those stresses at the section mn (normal and shearing) which are determined in strength of materials.

Some authors now use the following notations which are also employed in technical literature: the normal stresses

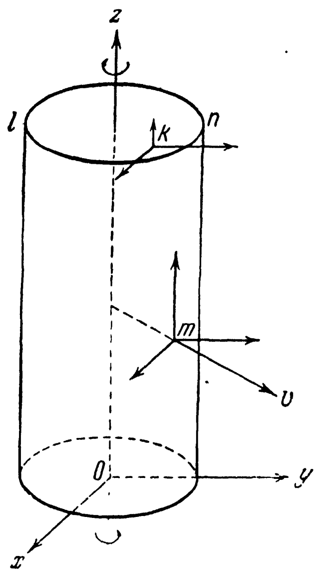

Fig. 7

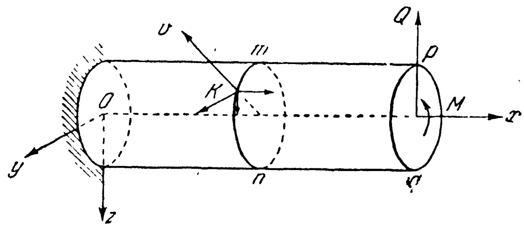

Fig. 8

on the areas normal to the axes OX, OY and OZ are denoted, respectively, by

$$\sigma_x, \ \sigma_y, \ \sigma_z;$$

the shearing stresses on the same areas are denoted by

$$\tau_{xy}, \ \tau_{yz}, \ \tau_{xz},$$

so that

$$X_y = \tau_{xy}; \quad Y_z = \tau_{yz}; \quad X_z = \tau_{xz}.$$

2. DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

Imagine that we isolate in a solid an infinitesimal element in the form of a parallelepiped with three pairs of faces parallel to the co-ordinate planes. Let the edges of this parallelepiped be dx, dy, dz (Fig. 9). Its volume is $d\tau = dx\,dy\,dz$. The action of the removed parts of the body upon the isolated element is replaced by forces. The stress on each face is resolved into three

components. Altogether there will be $6 \times 3 = 18$ stress components acting on all faces. These will be the external forces acting on our parallelepiped. In addition we shall assume that there exist the so-called body forces in the given body. These forces will be considered to be applied to the mass of the body; such is, for instance, the gravity force which will be dealt with later.

Let some body forces, which we also resolve into three components, X, Y, Z, be applied to unit mass of the body. It follows that these quantities have the dimension of acceleration. The body forces acting on the mass of the parallelepiped $\rho d\tau$ (ρ is the density of the body at a given point representing the mass per

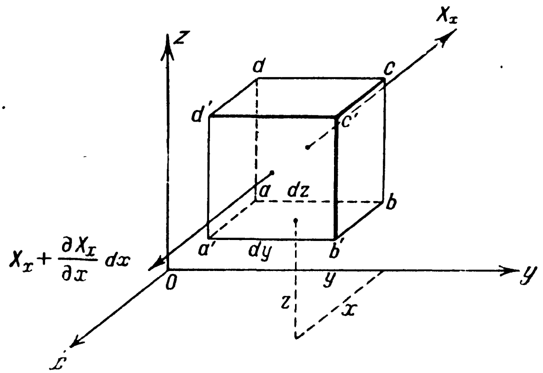

Fig. 9

unit of volume and having the dimension kg sec²/cm⁴) will then be

$$X\rho\, d\tau = X\rho\, dx\, dy\, dz;$$
$$Y\rho\, d\tau = Y\rho\, dx\, dy\, dz;$$
$$Z\rho\, d\tau = Z\rho\, dx\, dy\, dz.$$

The stresses set up in a solid by external forces vary from point to point, and they are therefore, generally speaking, functions of the co-ordinates of points

$$\left.\begin{aligned} X_x &= F_1(x,\ y,\ z); \\ Y_y &= F_2(x,\ y,\ z); \\ Z_z &= F_3(x,\ y,\ z). \\ &\ldots\ldots\ldots \end{aligned}\right\} \tag{1.1}$$

If the area $abcd$ of the element (Fig. 9) is acted upon by the stress $X_{-x} = -X_x$, the area $a'b'c'd'$ will be acted upon by the

stress $X_x + \frac{\partial X_x}{\partial x} dx$ since, in passing to the area $a'b'c'd'$, we change only one of the co-ordinates, namely x, in the first equality of (1.1). Thus, we can easily designate the stresses on all the areas bounding the parallelepiped.

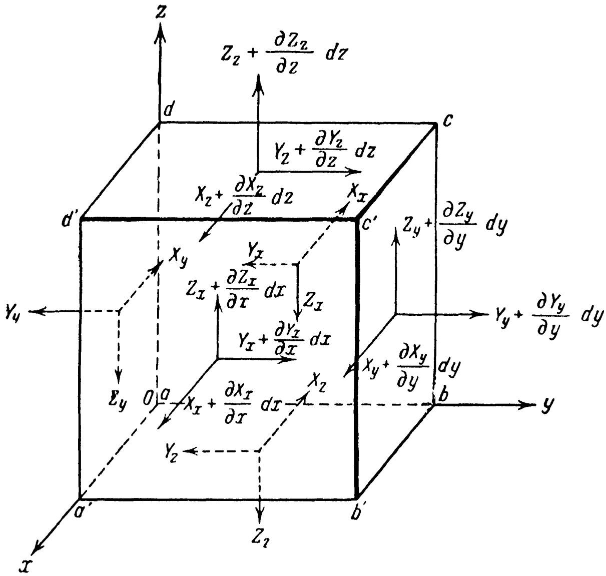

Fig. 10

Let us assume that the body under consideration is in equilibrium; the six equations of equilibrium of statics must then be satisfied for each particular element

$$\left.\begin{array}{ll} \Sigma X = 0, & \Sigma M_x = 0, \\ \Sigma Y = 0, & \Sigma M_y = 0, \\ \Sigma Z = 0, & \Sigma M_z = 0. \end{array}\right\} \qquad (1.2)$$

To begin with, examine the first group of these equations and, in particular, consider the first equation

$$\Sigma X = 0. \qquad (1.3)$$

Here we must take into account only those forces which give projections on the axis Ox; these will be the normal stresses on the lateral faces $abcd$ and $a'b'c'd'$ (Fig. 10) and the shearing stresses parallel to the axis Ox acting on the other faces.

Writing out the equation of equilibrium (1.3) we have

$$\left(X_x+\frac{\partial X_x}{\partial x}dx\right)dy\,dz-X_x dy\,dz+\left(X_y+\frac{\partial X_y}{\partial y}dy\right)dx\,dz-$$
$$-X_y\,dx\,dz+\left(X_z+\frac{\partial X_z}{\partial z}dz\right)dx\,dy-X_z\,dx\,dy+X\rho\,dx\,dy\,dz=0 \tag{1.4}$$
$$\left(=\rho\,dx\,dy\,dz\,\frac{\partial^2 u}{\partial t^2}\right).$$

If we consider the case of motion of particles of an elastic body (elastic vibrations), then ΣX does not vanish. According to Newton's second law, it is equal to the product of the mass of the element and the projection of its acceleration on the axis Ox; if the projections of the path (displacements) traversed by a given particle on the co-ordinate axes are denoted by u, v and w, the acceleration along the axis Ox is $\frac{\partial^2 u}{\partial t^2}$; on the right-hand side of equation (1.4) there appears, therefore, the expression

$$\rho\,d\tau\,\frac{\partial^2 u}{\partial t^2}=\rho\,dx\,dy\,dz\,\frac{\partial^2 u}{\partial t^2}$$

as indicated in parentheses.

After cancelling out like terms in (1.4) and dividing the resulting equation by $dx\,dy\,dz=d\tau$ we obtain it in the final form

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+X\rho=0\quad\left(=\rho\,\frac{\partial^2 u}{\partial t^2}\right).$$

In the same way we write out the other two equations

$$\Sigma Y=0,\quad \Sigma Z=0.$$

In the case of motion, the product of the mass of the element and acceleration along the respective axes should be substituted for zero on the right-hand side

$$\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+Y\rho=0\quad\left(=\rho\,\frac{\partial^2 v}{\partial t^2}\right);$$
$$\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+Z\rho=0\quad\left(=\rho\,\frac{\partial^2 w}{\partial t^2}\right).$$

We now proceed to write out the last three equations of equilibrium (1.2). Take, for instance, the equation $\Sigma M_x=0$. Accordingly, we retain in the drawing only the forces which can produce moments about the axis Ox, i.e., the ones that are normal to it. To simplify calculation we place the origin of co-ordinates at one of the vertices of the parallelepiped (Fig. 11). We note

that the moments of some of the forces shown in the drawing will be small quantities of the third order, others—of the fourth order. For instance, for the normal forces over the left-hand and right-hand faces we have the moment

$$\left(Y_y+\frac{\partial Y_y}{\partial y}\,dy\right)dx\,dz\,\frac{dz}{2}-Y_y\,dx\,dz\,\frac{dz}{2}.$$

This moment thus turns out to be a small quantity of the fourth order; of the same order will be the moments of the body forces

$$+Y\rho\,dx\,dy\,dz\,\frac{dz}{2};\quad -Z\rho\,dx\,dy\,dz\,\frac{dy}{2}.$$

Examining Fig. 11, we find only two forces producing the moment of the third order; they are indicated by thicker lines in

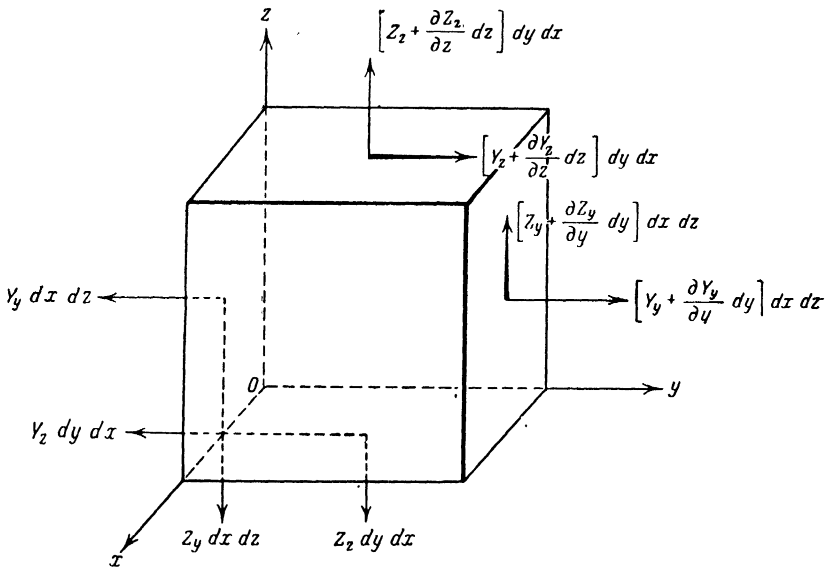

Fig. 11

the drawing. Equating to zero the sum of the moments of these forces about the axis Ox, we obtain

$$\left(Y_z+\frac{\partial Y_z}{\partial z}\,dz\right)dx\,dy\,dz-\left(Z_y+\frac{\partial Z_y}{\partial y}\,dy\right)dx\,dy\,dz=0;$$

dropping here the small quantities of the fourth order, we get

$$Y_z\,dx\,dy\,dz-Z_y\,dy\,dz\,dx=0,$$

or cancelling out, we have

$$Y_z = Z_y.$$

This is the *law of reciprocity (pairing, conjugation) of shearing stresses*. Accordingly, the other two equations give

$$Y_x = X_y; \quad Z_x = X_z.$$

Thus, the equilibrium conditions of statics have led us to three differential equations

$$\left.\begin{aligned}
\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X = 0 \quad \left(= \rho \frac{\partial^2 u}{\partial t^2}\right),\\
\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \rho Y = 0 \quad \left(= \rho \frac{\partial^2 v}{\partial t^2}\right),\\
\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho Z = 0 \quad \left(= \rho \frac{\partial^2 w}{\partial t^2}\right),
\end{aligned}\right\} \tag{1.5}$$

which contain nine functions of the co-ordinates of the point in question: X_x, X_y, X_z; Y_x, Y_y, Y_z; Z_x, Z_y and Z_z; however, the law of conjugation of shearing stresses

$$\left.\begin{aligned}
Y_z = Z_y,\\
Z_x = X_z,\\
X_y = Y_x
\end{aligned}\right\} \tag{1.6}$$

shows that three of these nine functions are pairwise equal to three others. Consequently, we have three differential equations (1.5) containing six unknown functions

$$\left.\begin{aligned}
X_x = F_1(x, y, z),\\
Y_y = F_2(x, y, z),\\
Z_z = F_3(x, y, z),\\
Y_z = F_4(x, y, z),\\
Z_x = F_5(x, y, z),\\
X_y = F_6(x, y, z).
\end{aligned}\right\} \tag{1.7}$$

In the case of elastic vibrations the right-hand members of these equations depend on the time t as well.

Since the number of unknowns (1.7) exceeds the number of equations (1.5), we conclude that the problem of the theory of elasticity is *statically indeterminate*. The lacking equations can be obtained only by studying the conditions of deformation and by taking into consideration the physical properties of a given elastic body. In fact, up to this point we have not been con-

cerned with these properties when isolating in a body an element for which equations (1.5) have been derived; the state of stress in a body, however, depends undoubtedly on these properties.

The method by which we have derived equations (1.5) is often employed in the elementary theory of bending of bars, for instance, to obtain the relations between bending moment, shearing force and load:

$$\frac{dQ}{dx} = -q; \quad \frac{dM}{dx} = Q,$$

or to calculate shearing stresses in bending [1].

For that purpose we isolate in a bar and balance an infinitely thin layer between two cross sections, i.e., an element which is infinitesimal in one direction; here, however, we obtain the conditions of equilibrium of an element which is infinitesimal in all three dimensions.

3. STRESSES ON AREAS INCLINED TO THE CO-ORDINATE PLANES. SURFACE CONDITIONS

The equations of equilibrium (1.5) correlate stresses over areas parallel to the co-ordinate planes.

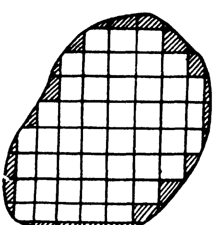

Fig. 12

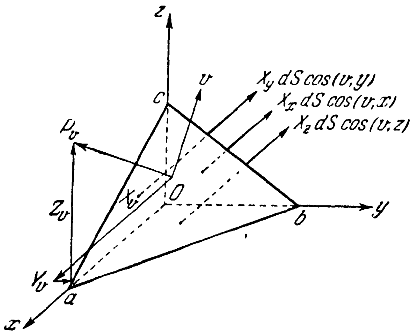

Fig. 13

It is sometimes necessary, however, to know stresses over areas oriented in a different manner; on the other hand, if the whole elastic body is divided into elements by planes parallel to the co-ordinate planes, it will generally be impossible to isolate elements in the form of parallelepipeds at the surface of the body (Fig. 12). The inclined faces thus obtained at the surface will

[1] These relations are derived in all courses on strength of materials.

obviously be acted upon by the external forces (loads) applied to the given body. The foregoing considerations also induce us to derive relations between stresses over three areas parallel to the co-ordinate planes and stresses over an area inclined arbitrarily to these planes.

To obtain such relations, consider (Fig. 13) an infinitesimal element isolated in a given solid by three planes parallel to the co-ordinate planes and by the fourth plane intersecting all the three axes, i.e., consider an element *Oabc* in the form of a tetrahedron, or a triangular pyramid. Let the area of its face *abc* be denoted by dS and the outward normal to it by v; then the areas of the other faces will be found as the projections of the area of the face *abc* on the co-ordinate planes

$$\begin{aligned} \text{area } abc &= dS; \\ \text{area } Oab &= dS\cos(vz); \\ \text{area } Obc &= dS\cos(vx); \\ \text{area } Oac &= dS\cos(vy)\,^{1}. \end{aligned}$$

Let X_v, Y_v, Z_v be the projections on the co-ordinate axes of the total stress P_v acting on the area *abc*. The first condition of equilibrium of the isolated tetrahedron $\Sigma X = 0$ gives then

$$\Sigma X = X_v dS - X_x dS\cos(vx) - X_y dS\cos(vy) - X_z dS\cos(vz) = 0.$$

Cancelling out dS, we have

$$X_v = X_x\cos(vx) + X_y\cos(vy) + X_z\cos(vz).$$

Similarly, the other two conditions of equilibrium give

$$\left.\begin{aligned} Y_v &= Y_x\cos(vx) + Y_y\cos(vy) + Y_z\cos(vz); \\ Z_v &= Z_x\cos(vx) + Z_y\cos(vy) + Z_z\cos(vz). \end{aligned}\right\} \quad (1.8)$$

Let us introduce an abbreviated notation of the direction cosines of the outward normal v to the area

$$\cos(vx) = l; \quad \cos(vy) = m; \quad \cos(vz) = n. \qquad (1.9)$$

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[1] Here x, y, z in parentheses denote the directions parallel to the co-ordinate axes Ox, Oy, Oz, respectively.

Then equations (1.8) will be written down in the form

$$\left.\begin{aligned} X_v &= X_x l + X_y m + X_z n; \\ Y_v &= Y_x l + Y_y m + Y_z n; \\ Z_v &= Z_x l + Z_y m + Z_z n, \end{aligned}\right\} \qquad (1.8a)$$

which will be frequently used in what follows.

The body forces are not involved in these equations since they are small quantities of the third order, for instance:

$\rho X\, d\tau = \rho X \frac{dx\, dy\, dz}{6}$, while all terms of the first equation of (1.8) in its initial form are small quantities of the second order.

We assume that the area of the face *abc* is made to approach zero as a limit; equations (1.8) give then the relation between the stresses at a point O on an *oblique area* with outward normal v and on three areas parallel to the co-ordinate planes. If we cut out a tetrahedron *Oabc* at the surface with the face *abc* belonging to the surface, then X_v, Y_v, Z_v are the stress components due to the external forces (loads on the given body) applied on the surface. Equations (1.8) give then the relation between the external load and internal forces. In this case they are called the *conditions on the surface* of a body and appear to be very closely connected with the differential equations of equilibrium (1.5); in fact, if functions (1.7) are such that equations (1.5) and surface conditions (1.8) are satisfied at all points of the body and on its surface, this guarantees the equilibrium of all elements (parallelepipeds and tetrahedrons) into which the given body has been divided; hence, the equilibrium of the body as a whole will be secured. The mathematical meaning of this conclusion is that equations (1.5) and boundary conditions (1.8) must be considered simultaneously, for equations (1.5) cannot have any definite meaning unless conditions (1.8) involving the external load on a body are given.

Thus, the body as a whole will be in equilibrium if the equations of equilibrium (1.5) are satisfied and the surface loads are specified; in other words, we have proved the sufficiency of equations (1.5) and boundary conditions (1.8) for the equilibrium of a body under the action of the given external forces. One can prove the necessity of equations (1.5). Indeed, if a body is in equilibrium, six conditions of equilibrium (1.2) are satisfied for any part isolated in it by a closed surface S. The first of them is written out as

$$\int\limits_{(S)} X_v\, dS + \int\limits_{(\tau)} X\rho\, d\tau = 0.$$

Here the first integral is extended over the surface of the body, and the second integral over its volume. However, the first integral can also be transformed into a volume integral by the use of the Green-Ostrogradsky formula [1]

$$\int_{(S)} X_v\,dS = \int_{(S)} [X_x \cos(vx) + X_y \cos(vy) + X_z \cos(vz)]\,dS =$$
$$= \int_{(\tau)} \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) d\tau. \quad (1.10)$$

On this basis the preceding equation of equilibrium may be written down as

$$\int_{(\tau)} \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X\right) d\tau = 0.$$

The integral on the left-hand side must vanish for any volume τ isolated in the body, but this is possible only if the integrand is identically equal to zero at all points of the volume τ:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X = 0.$$

This proves the necessity of the first of equations (1.5). In the same way one can prove the necessity of the other two equations of (1.5) and equations (1.6).

4. ANALYSIS OF THE STATE OF STRESS AT A GIVEN POINT IN A BODY. PRINCIPAL AREAS AND PRINCIPAL STRESSES

Let us suppose that an element *Oabc* has been cut out within an elastic body and that X_v, Y_v, Z_v are the projections of the total stress P_v over the oblique area *abc* on the axes of a random co-ordinate system *Oxyz* (Fig. 13). Knowing the projections of the total stress (X_v, Y_v, Z_v), we readily find the normal and shearing stresses on the same area *abc*.

---

[1] The Green-Ostrogradsky formula has the form

$$\int_{(\tau)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) d\tau = \int_{(S)} [P \cos(vx) + Q \cos(vy) + R \cos(vz)]\,dS,$$

where P, Q, R are functions of x, y, z, being continuous on the closed surface S; their partial derivatives must exist and must be continuous throughout the volume τ bounded by the surface S; v is the outward normal to the surface S at a certain point (x, y, z).

For this purpose we take the normal v to the area to be one of new co-ordinate axes. The other two axes u and w will be chosen in the plane abc (Fig. 14). We denote the cosines of the angles

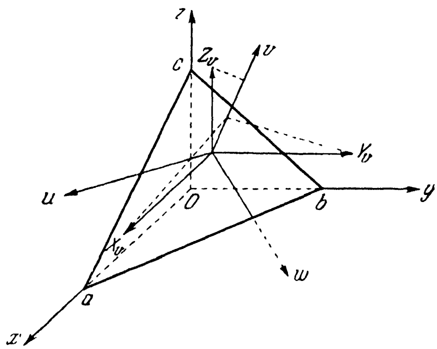

Fig. 14

between the old and new axes according to the scheme

	x	y	z
u	l_1	m_1	n_1
v	l_2	m_2	n_2
w	l_3	m_3	n_3

Now equations (1.8), similarly to (1.8a), may be rewritten more compactly

$$\left.\begin{array}{ll} X_v = X_x l_2 + X_y m_2 + X_z n_2 & \vert\, l_2 \\ Y_v = Y_x l_2 + Y_y m_2 + Y_z n_2 & \vert\, m_2 \\ Z_v = Z_x l_2 + Z_y m_2 + Z_z n_2 & \vert\, n_2 \end{array}\right\}. \tag{1.11}$$

Obviously (Fig. 14), the normal stress on the area abc will be obtained as the sum of the projections of stresses X_v, Y_v, Z_v on the normal v.

Designating the normal stress, according to the system accepted above, by V_v, we get

$$\begin{aligned} V_v &= X_v l_2 + Y_v m_2 + Z_v n_2 = \\ &= X_x l_2^2 + Y_y m_2^2 + Z_z n_2^2 + 2Y_z m_2 n_2 + 2Z_x n_2 l_2 + 2X_y l_2 m_2. \end{aligned} \tag{1.12}$$

Similarly we obtain the shearing stresses projecting X_v, Y_v, Z_v on the axes u and w, respectively:

$$\left.\begin{aligned} U_v &= X_v l_1 + Y_v m_1 + Z_v n_1 = X_x l_1 l_2 + Y_y m_1 m_2 + Z_z n_1 n_2 + \\ &+ Y_z (m_1 n_2 + m_2 n_1) + Z_x (n_2 l_1 + n_1 l_2) + X_y (l_1 m_2 + l_2 m_1); \\ W_v &= X_v l_3 + Y_v m_3 + Z_v n_3 = X_x l_2 l_3 + Y_y m_2 m_3 + Z_z n_2 n_3 + \\ &+ Y_z (m_3 n_2 + m_2 n_3) + Z_x (n_3 l_2 + n_2 l_3) + X_y (m_2 l_3 + m_3 l_2). \end{aligned}\right\} \qquad (1.13)$$

Formulas (1.12) and (1.13) are considerably simplified in the case of the so-called state of plane stress when all forces are parallel to a single plane and distributed uniformly in a direction perpendicular to this plane.

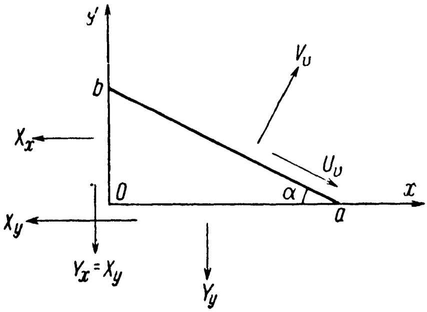

Fig. 15

In fact, if all forces are located in the plane Oxy, one can isolate a triangular prism Oab in place of the tetrahedron $Oabc$ (Fig. 15). In formulas (1.12) and (1.13) X_z, Y_z, Z_z disappear; these formulas then become

$$\left.\begin{aligned} V_v &= X_x l_2 + Y_y m_2^2 + 2X_y l_2 m_2, \\ U_v &= X_x l_1 l_2 + Y_y m_1 m_2 + X_y (l_1 m_2 + l_2 m_1). \end{aligned}\right\} \qquad (1.14)$$

The angles (ux), (uy), (vx) and (vy) are indicated in scheme (a), and the respective cosines — in scheme (b)

	x	y
u	α	$\frac{\pi}{2}+\alpha$
v	$\frac{\pi}{2}-\alpha$	α

(a)

	x	y
u	$l_1 = \cos\alpha$	$m_1 = -\sin\alpha$
v	$l_2 = \sin\alpha$	$m_2 = \cos\alpha$

(b)

On the basis of scheme (b) equations (1.14) take the form

$$\left.\begin{aligned} V_v &= X_x \sin^2\alpha + Y_y \cos^2\alpha + 2X_y \sin\alpha\cos\alpha, \\ U_v &= (X_x - Y_y)\sin\alpha\cos\alpha + X_y(\cos^2\alpha - \sin^2\alpha). \end{aligned}\right\} \tag{1.15}$$

They are known in this form from strength of materials; equations (1.15) can, of course, be derived directly from the consideration of Fig. 15.

Formulas (1.12) and (1.13) state the important proposition that if the stresses over three mutually perpendicular elementary areas at a point O are known

$$\left.\begin{matrix} X_x, & X_y, & X_z, \\ Y_x, & Y_y, & Y_z, \\ Z_x, & Z_y, & Z_z, \end{matrix}\right\} \tag{1.16}$$

we can calculate all three components of stress

$$V_v, \ U_v, \ W_v$$

over any fourth area (at the point O) determined by its outward normal v; in view of this the quantities (1.16) are sometimes called elements of the state of stress in a body. One can proceed further and construct two more areas with outward normals u and w at the point O; the stresses over these areas will be expressed by formulas of the same form as (1.12) and (1.13), for instance

$$U_u = X_x l_1^2 + Y_y m_1^2 + Z_z n_1^2 + 2Y_z m_1 n_1 + 2Z_x n_1 l_1 + 2X_y l_1 m_1.$$

After this has been done, we obtain nine formulas [three of the type (1.12) and six of the type (1.13)] expressing the stress components over the areas of the trihedral $Ouvw$

$$\left\{\begin{matrix} U_u & V_u & W_u \\ U_v & V_v & W_v \\ U_w & V_w & W_w \end{matrix}\right\} \tag{1.17}$$

in terms of stresses (1.16) over the areas of the former trihedral $Oxyz$.

Nine components (1.16) or (1.17) define the *stress tensor* and from this point of view they are called *components of the stress tensor*. The formulas of the type (1.12) and (1.13) (as we have said, there are nine of them) determine the transformation of the tensor from one co-ordinate system to another. The stress tensor (1.16) is symmetric, since the components symmetric with

respect to the principal diagonal (X_x, Y_y, Z_z) are equal to each other by virtue of (1.16); this property is obviously maintained for other systems of co-ordinates as well [1].

5. STRESS DISTRIBUTION AT A GIVEN POINT. CAUCHY'S STRESS SURFACE; INVARIANTS OF THE STRESS TENSOR. LAMÉ'S ELLIPSOID

Let us now turn to the very important question of how stresses are distributed over all the possible areas passing through a given point in a body; for this purpose we shall apply a very elegant and visual method devised by the founder of the theory of elasticity, A. L. Cauchy. We place the origin of co-ordinates at some chosen point M in a body, pass an elementary area with outward normal v, and along this normal construct a vector of length p which as yet is left undefined. The co-ordinates of the end of this vector are

$$\xi = pl; \quad \eta = pm; \quad \zeta = pn, \tag{1.18}$$

where l, m, n are the cosines of the angles that the normal makes with the co-ordinate axes. The normal stress on the area in question according to (1.12) is

$$V_v = X_x l^2 + Y_y m^2 + Z_z n^2 + 2Y_z mn + 2Z_x nl + 2X_y lm. \tag{1.19}$$

We determine l, m, n from the preceding equalities and substitute them in (1.19); multiplied by p^2, the equation reduces to

$$X_x \xi^2 + Y_y \eta^2 + Z_z \zeta^2 + 2Y_z \eta\zeta + 2Z_x \zeta\xi + 2X_y \xi\eta = p^2 V_v. \tag{1.20}$$

We now adjust the length of the vector p so that the right-hand member of this equation will always remain constant

$$p^2 V_v = \pm c^2;$$

the + sign should be taken on the right-hand side if $V_v > 0$, i.e., if the normal stress on the area is tensile, and the — sign if it is compressive. Then

$$p = \sqrt{\frac{\pm c^2}{V_v}}, \quad \text{or} \quad V_v = \pm \frac{c^2}{p^2}. \tag{1.21}$$

Introducing the notation

$$\Phi(\xi, \eta, \zeta) = X_x \xi^2 + Y_y \eta^2 + Z_z \zeta^2 + 2Y_z \eta\zeta + 2Z_x \zeta\xi + 3X_y \xi\eta, \tag{1.22}$$

[1] For further details in regard to this tensor see N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Groningen, P. Noordhoff, 1953.

we write down equation (1.20) as

$$\Phi(\xi, \eta, \zeta) = \pm c_2. \tag{1.23}$$

It represents a surface of the second order which is called the stress surface or Cauchy's quadric; this surface has the centre at the origin [1] and it may, therefore, be (a) an ellipsoid, (b) a hyperboloid of one sheet, or (c) a hyperboloid of two sheets; the ends of the vectors **p** lie on this surface.

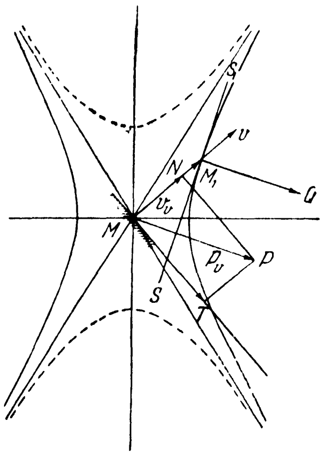

Fig. 16

Let us suppose, for the sake of definiteness, that the quadric is a hyperboloid of one sheet and that it has been constructed; to simplify the drawing, Fig. 16 shows its section by one of the principal planes on the assumption that the outward normal Mv to the area lies in this plane. Taking the area and drawing the outward normal Mv to it we find the length MM_1 of the vector **p**; knowing it, we determine the normal stress $V_v = \overline{MN}$ by the second of formulas (1.21).

Let us now show that we can easily find the total stress P_v on the given area as well. In fact, we take partial derivatives of function (1.22) and make use of (1.18) and (1.8a). We obtain

$$\frac{\partial\Phi}{\partial\xi} = 2(X_x\xi + X_y\eta + X_z\zeta) = 2p(X_xl + X_ym + X_zn) = 2pX_v;$$

$$\frac{\partial\Phi}{\partial\eta} = 2(Y_x\xi + Y_y\eta + Y_z\zeta) = 2p(Y_xl + Y_ym + Y_zn) = 2pY_v;$$

$$\frac{\partial\Phi}{\partial\zeta} = 2(Z_x\xi + Z_y\eta + Z_z\zeta) = 2p(Z_xl + Z_ym + Z_zn) = 2pZ_v.$$

It is known that these partial derivatives are proportional to the cosines of the angles that the normal M_1Q to the surface makes with the co-ordinate axes; on this basis the last equalities show that the cosines of the direction angles of the normal to the surface are proportional to the projections X_v, Y_v, Z_v of the total stress over the area concerned on the co-ordinate axes. Hence the conclusion: the total stress MP on the area is perpendicular to the tangent plane SS_1 to the surface; knowing its direction, we obtain

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[1] Since there are no terms of the first degree in ξ, η, ζ in equation (1.23).

its magnitude $P_v = \overline{MP}$ by drawing $NP \perp MN$. Now, of course, we can easily find the shearing stress MT. Thus, Cauchy's quadric enables the stress distribution at the given point M of the body to be fully investigated.

It is known from analytic geometry that by rotating the co-ordinate axes the equation of the surface of the second order (1.23) can be transformed so as to eliminate the terms containing products of co-ordinates in a new co-ordinate system (u, v, w). Then, we shall obviously have

$$U_v = 0; \quad V_w = 0; \quad W_u = 0.$$

Hence, the shearing stresses will vanish on three mutually perpendicular areas normal to the new axes (u, v, w). These three areas are called *principal*; the normal stresses on them U_u, V_v, W_w (they will also be the total stresses) are called *principal stresses* at the given point M in the body. We denote them more compactly

$$U_u = N_1; \quad V_v = N_2; \quad W_w = N_3.$$

Then the equation of Cauchy's quadric referred to the new axes is

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = \pm c^2. \tag{1.24}$$

The final form of this surface depends on the signs of the principal stresses N_1, N_2, N_3 and on the associated sign of the right-hand member of the equation. Consider two fundamental cases: (1) all three principal stresses are of the same sign; (2) the signs of the principal stresses are different.

Case 1. Let us assume that $N_1 > 0$, $N_2 > 0$, $N_3 > 0$, i.e., all the principal stresses are tensile. Then the plus sign should apparently be chosen in the right-hand member of (1.24), and it can be rewritten as

$$\frac{\xi^2}{\left(\frac{c^2}{N_1}\right)} + \frac{\eta^2}{\left(\frac{c^2}{N_2}\right)} + \frac{\zeta^2}{\left(\frac{c^2}{N_3}\right)} = 1. \tag{1.25}$$

This is the equation of an ellipsoid with the semi-axes

$$\bar{a} = \frac{c}{\sqrt{N_1}}; \quad \bar{b} = \frac{c}{\sqrt{N_2}}; \quad \bar{c} = \frac{c}{\sqrt{N_3}}.$$

One sees from formula (1.21) that in this case $V_v = +\frac{c^2}{p^2} > 0$, i.e., the normal stresses on all areas passing through the point in question are positive and consequently tensile. If all three principal stresses are compressive: $N_1 < 0$, $N_2 < 0$, $N_3 < 0$, the minus sign should be taken before c^2 in (1.24); this equation once again

gives an ellipsoid, but according to formula (1.21) we get

$$V_v = -\frac{c^2}{p^2} < 0,$$

and the normal stresses on all areas are compressive.

Case 2. Let us assume that $N_1 > 0$, $N_2 > 0$, $N_3 < 0$, i.e., two of the principal stresses are tensile and the third is compressive. In this case equation (1.24) has to be written in two ways:

$$N_1\xi^2 + N_2\eta^2 - |N_3|\zeta^2 = +c^2,$$
$$N_1\xi^2 + N_2\eta^2 - |N_3|\zeta^2 = -c^2,$$

where $|N_3|$ denotes the absolute value of the compressive principal stress. Just as in the first case, we write these equations in the form

$$\frac{\xi^2}{\left(\frac{c^2}{N_1}\right)} + \frac{\eta^2}{\left(\frac{c^2}{N_2}\right)} - \frac{\zeta^2}{\left(\frac{c^2}{|N^3|}\right)} = 1, \tag{1.26}$$

$$-\frac{\xi^2}{\left(\frac{c^2}{N_1}\right)} - \frac{\eta^2}{\left(\frac{c^2}{N_2}\right)} + \frac{\zeta^2}{\left(\frac{c^2}{|N_3|}\right)} = 1. \tag{1.27}$$

The first of these equations gives a hyperboloid of one sheet, and the second — a hyperboloid of two sheets. We add an intermediate case, setting $c^2 = 0$:

$$N_1\xi^2 + N_2\eta^2 - |N_3|\zeta^2 = 0. \tag{1.28}$$

This equation gives the asymptotic cone separating the above two hyperboloids and tending to approach them at infinity (Fig. 16).

If the end of the vector representing the normal stress on the area appears to be on the hyperboloid of one sheet (1.26), this stress is positive, i.e., tensile; if, on the other hand, it appears to be on the hyperboloid of two sheets, the stress is compressive. In the intermediate case it may be directed along the generator of the asymptotic cone (1.28); in this case the length of the vector becomes infinite and according to (1.21) $V_v = 0$. Hence, on areas normal to the generators of the asymptotic cone there act only shearing stresses.

If a given area *abc* (Fig. 14) is a principal area, the total stress P_v acting on it is directed along the normal v and is a principal stress: we denote it by N; its projections on the coordinate axes are

$$X_v = Nl; \quad Y_v = Nm; \quad Z_v = Nn.$$

We substitute these values in equations (1.21) and write them down as

$$\left.\begin{aligned}(X_x - N)\,l + X_y m + X_z n &= 0,\\ Y_x l + (Y_y - N)\,m + Y_z n &= 0,\\ Z_x l + Z_y m + (Z_z - N)\,n &= 0.\end{aligned}\right\} \tag{1.29}$$

Adding here the basic relation among the direction cosines of the normal v

$$l^2 + m^2 + n^2 = 1, \tag{1.30}$$

we have four equations for the determination of the principal stress N and the corresponding principal area, i.e., the direction cosines of its normal, l, m, n. The procedure of solution will be as follows: the system of homogeneous equations (1.29) does not admit a trivial solution

$$l = m = n = 0,$$

because it is contrary to (1.30); however, for the existence of other solutions of this system (in which at least one of the cosines l, m, n has a value different from zero) it is necessary that its determinant should be equal to zero. Thus, we come to the condition

$$\begin{vmatrix} (X_x - N) & X_y & X_z \\ Y_x & (Y_y - N) & Y_z \\ Z_x & Z_y & (Z_z - N) \end{vmatrix} = 0. \tag{1.31}$$

Writing out the determinant in the left-hand member, we obtain the cubic equation

$$N^3 - \Theta N^2 + \mathrm{H} N - \Lambda = 0, \tag{1.32}$$

in which the coefficients have the following values:

$$\left.\begin{aligned}\Theta &= X_x + Y_y + Z_z;\\ \mathrm{H} &= \begin{vmatrix} X_x X_y \\ X_y Y_y \end{vmatrix} + \begin{vmatrix} X_x X_z \\ X_z Z_z \end{vmatrix} + \begin{vmatrix} Y_y Y_z \\ Y_z Z_z \end{vmatrix};\\ \Lambda &= \begin{vmatrix} X_x X_y X_z \\ Y_x Y_y Y_z \\ Z_x Z_y Z_z \end{vmatrix}.\end{aligned}\right\} \tag{1.33}$$

All three roots of equation (1.32) are real [1]; they give the values of the three principal stresses N_1, N_2, N_3. Substituting any

[1] This is proved in analytic geometry in investigating the principal axes of the central surface of the second order.

of these values of N in equations (1.29) and using two of them[1], we find the quantities l_i', m_i', n_i' proportional to the direction cosines of the normal to the unknown principal area

$$l_i' = \lambda l_i; \quad m_i' = \lambda m_i; \quad n_i' = \lambda n_i.$$

Substituting this in (1.30), we find the factor λ and, consequently, the cosines themselves. Thus, the directions of the principal areas will be determined.

Another point: the roots of equation (1.32) must not depend on the system of co-ordinates x, y, z; consequently, the coefficients of this equation do not depend on the choice of a system of co-ordinates either. Hence we conclude that formulas (1.33) give three functions of the components of the stress tensor (1.16) which are invariants under a transformation of co-ordinates. Of particular significance is the first of them, the linear invariant

$$\Theta = X_x + Y_y + Z_z, \tag{1.34}$$

as we shall see in Chapter III.

If the principal areas at a given point have been found, then, along with Cauchy's quadric, it is possible to indicate another geometrical representation of stress distribution, the one proposed by G. Lamé.

Let us assume that the co-ordinate planes Oxy, Oyz and Ozx coincide with the principal areas at a given point and, consequently, on these areas: $Y_z = Z_y = 0$; $Z_x = X_z = 0$; $X_y = Y_x = 0$; $X_x = N_1$; $Y_y = N_2$; $Z_z = N_3$.

Then equations (1.8a) are simplified and take the form

$$\left.\begin{aligned} X_v &= N_1 l, \\ Y_v &= N_2 m, \\ Z_v &= N_3 n. \end{aligned}\right\} \tag{1.35}$$

Relationships (1.35) may be represented geometrically. To this end, we lay off from the given point O (Fig. 17) a vector $\overline{OP}$ equal to the total stress on the chosen area with outward normal v; the co-ordinates of the end of this vector are

$$\left.\begin{aligned} x &= X_v; \\ y &= Y_v; \\ z &= Z_v. \end{aligned}\right\} \tag{1.36}$$

[1] The third equation will be a consequence of the other two, according to condition (1.31).

As the inclination of the area is changed, the point P describes a certain surface that appears to be an ellipsoid; indeed, on the basis of (1.35) and (1.36) we have

$$x = N_1 l; \quad y = N_2 m; \quad z = N_3 n;$$

hence

$$\frac{x^2}{N_1^2} + \frac{y^2}{N_2^2} + \frac{z^2}{N_3^2} = l^2 + m^2 + n^2 = 1; \tag{1.37}$$

this is the equation of an ellipsoid referred to the principal axes. This ellipsoid is called the stress ellipsoid or Lamé's ellipsoid. One of its three semi-axes is the longest, another the shortest, the third is half-way between them; consequently, the three principal stresses are maximum, minimum and minimax, respectively.

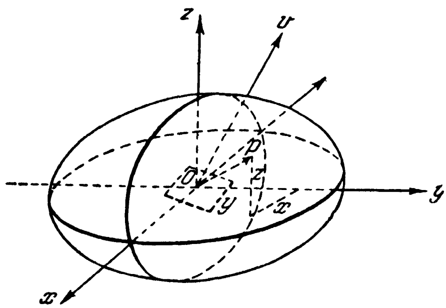

Fig. 17

If two of the principal stresses are equal (for instance, $N_1 = N_2$), Lamé's ellipsoid is an ellipsoid of revolution and the state of stress at a given point is symmetrical with respect to the third principal axis Oz. If all the principal stresses are equal, $N_1 = N_2 = N_3$, Lamé's ellipsoid becomes a sphere and all the areas at the given point are principal and the stresses on them are equal; this will be the case, for instance, under all-round compression or tension.

6. MAXIMUM SHEARING STRESSES

Let us now turn to the determination of the maximum shearing stresses. To simplify the investigation, we take again the axes of co-ordinates $Oxyz$ in the direction of the normals to the principal areas (i.e., along the principal stresses).

We choose an arbitrary area with outward normal v determined by the cosines l, m, n; the total stress on it P_v, according to

(1.35), has the projections

$$X_v = N_1 l; \quad Y_v = N_2 m; \quad Z_v = N_3 n.$$

The normal stress on this area V_v, according to (1.19), is

$$V_v = N_1 l^2 + N_2 m^2 + N_3 n^2.$$

If we denote by T_v the total shearing stress on the area concerned, we have

$$P_v^2 = V_v^2 + T_v^2,$$

whence we find

$$T_v^2 = P_v^2 - V_v^2 = (X_v^2 + Y_v^2 + Z_v^2) - V_v^2 =$$
$$= (N_1^2 l^2 + N_2^2 m^2 + N_3^2 n^2) - (N_1 l^2 + N_2 m^2 + N_3 n^2)^2. \quad (1.38)$$

With rotating area the stress T_v varies, being a function of two variables, say l and m, since according to (1.30)

$$n^2 = 1 - l^2 - m^2;$$

taking this into account, the preceding expression can be written as

$$T_v^2 = (N_1^2 - N_3^2) l^2 + (N_2^2 - N_3^2) m^2 + N_3^2 -$$
$$- [(N_1 - N_3) l^2 + (N_2 - N_3) m^2 + N_3]^2. \quad (1.39)$$

To obtain $\frac{\max}{\min} T_v$ we equate to zero the partial derivatives of T_v with respect to l and m; it is more convenient, however, to set up other conditions

$$\frac{\partial T_v^2}{\partial l} = 2T_v \frac{\partial T_v}{\partial l} = 0;$$

$$\frac{\partial T_v^2}{\partial m} = 2T_v \frac{\partial T_v}{\partial m} = 0.$$

It may be well to note that, in addition to the required solutions, we shall then obtain the extraneous solution $T_v = 0$ which will obviously give a principal area; this solution will easily be found and dropped later. Writing out the preceding conditions and using (1.39), we obtain two equations for the determination of l and m

$$(N_1^2 - N_3^2) l - 2[(N_1 - N_3) l^2 + (N_2 - N_3) m^2 + N_3](N_1 - N_3) l = 0;$$

$$(N_2^2 - N_3^2) m - 2[(N_1 - N_3) l^2 + (N_2 - N_3) m^2 + N_3](N_2 - N_3) m = 0.$$

Consider the most general case when N_1, N_2 and N_3 are all different; then the preceding equations can be divided through by the differences of the principal stresses entering in them to

represent these equations, after some manipulation, as

$$\left.\begin{aligned}\{N_1-N_3-2\,[(N_1-N_3)\,l^2+(N_2-N_3)\,m^2]\}\,l=0;\\ \{N_2-N_3-2\,[(N_1-N_3)\,l^2+(N_2-N_3)\,m^2]\}\,m=0.\end{aligned}\right\}\quad(1.40)$$

We have two equations of the third degree in l and m; accordingly, we shall obtain three solutions that are easy to find. The first and the simplest, in which

$$l=m=0,\quad n=1,$$

must be dropped since it gives a principal area lying in the plane Oxy. It will therefore be necessary to consider three cases:

$$(1)\ l\neq 0;\ m=0;\quad (2)\ l=0;\ m\neq 0;\quad (3)\ l\neq 0;\quad m\neq 0.$$

The third case is, however, impossible since then, cancelling l and m out of equations (1.40), respectively, and subtracting the resulting equations one from the other, we immediately obtain

$$N_1=N_2,$$

which is contrary to the initial assumption. In the first case ($l\neq 0$; $m=0$) the second of equations (1.40) is satisfied, and the first equation, after the cancelling out of l, reduces to

$$(N_1-N_3)(1-2l^2)=0,\qquad(1.41)$$

whence

$$l=\pm\frac{1}{\sqrt{2}};\quad m=0;\quad n=\pm\frac{}{\sqrt{2}}.$$

In the second case, from the second equation of (1.40) we obtain similarly

$$l=0;\quad m=\pm\frac{1}{\sqrt{2}};\quad n=\pm\frac{1}{\sqrt{2}}.$$

If, at the outset, we had eliminated, say, m instead of n from (1.38) and repeated the analysis, we would have obtained additionally one more solution

$$l=\pm\frac{1}{\sqrt{2}};\quad m=\pm\frac{1}{\sqrt{2}};\quad n=0.$$

Each of these three solutions determines two areas passing through one of the co-ordinate axes and inclined to the other two at angles of 45° and 135°.

Substituting the first solution in (1.38), we obtain the required maximum (or minimum) value of the shearing stress

$$T_v^2=\frac{N_1^2+N_3^2}{2}-\left(\frac{N_1+N_3}{2}\right)^2=\left(\frac{N_1-N_3}{2}\right)^2;$$

replacing the notation T_v by T_2, we find

$$\left.\begin{aligned} T_2 &= \pm\frac{N_1 - N_3}{2}; \\ &\text{the remaining two solutions give similarly} \\ T_1 &= \pm\frac{N_2 - N_3}{2}; \\ T_3 &= \pm\frac{N_1 - N_2}{2}. \end{aligned}\right\} \qquad (1.42)$$

We draw the final conclusion from the analysis carried out. The three mutually perpendicular principal areas at a point M

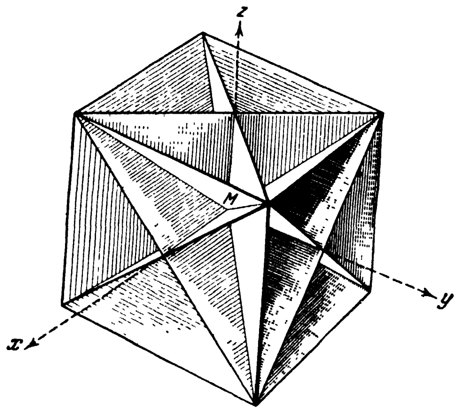

Fig. 18

form a trihedral. If (Fig. 18) the axes Mx, My, Mz represent the lines of their intersection, three pairs of areas on which the maximum shearing stresses occur pass through these axes and bisect the dihedral angles between the principal areas.

The maximum shearing stresses, according to (1.42), are equal to half the difference between the principal stresses over two areas intersecting along that of the axes x, y, z through which a given area of max T passes.

If

$$N_1 > N_2 > N_3,$$

the greatest of stresses (1.42) will be

$$T_2 = \pm\frac{N_1 - N_3}{2}.$$

If, for instance,

$$N_1 = N_2 > N_2,$$

Cauchy's quadric and Lamé's ellipsoid will be surfaces of revolution about the axis Mz; all the areas passing through the axis Mz (there is an infinite number of them) will be principal. The areas on which max $T=T_2$ are also infinite in number; they are tangent to a circular cone with the axis Mz and the vertex angle of 90°.

*7. OCTAHEDRAL AREAS AND OCTAHEDRAL STRESSES[1]

Formulas (1.19) and (1.38) make it possible to determine the normal and shearing stresses over any area at a given point if the principal areas and principal stresses are known. Keeping this in mind, let us find stresses over areas equally inclined to the principal areas. In eight octants of the co-ordinate system we can thus construct eight areas which will form an octahedron; they are called octahedral; the stresses on them are also called *octahedral.*

The direction cosines of the normals to these areas are

$$l=\pm\frac{1}{\sqrt{3}};\quad m=\pm\frac{1}{\sqrt{3}};\quad n=\pm\frac{1}{\sqrt{3}}.$$

Substituting these values in (1.19) and (1.38), we have

$$V_v=N_0=\frac{1}{3}(N_1+N_2+N_3)=\frac{1}{3}(X_x+Y_y+Z_z); \tag{1.43}$$

the square of the shearing octahedral stress is

$$T_0^2=\frac{1}{9}(2N_1^2+2N_2^2+2N_3^2-2N_1N_2-2N_2N_3-2N_3N_1)=$$
$$=\frac{1}{9}[(N_1-N_2)^2+(N_2-N_3)^2+(N_3-N_1)^2]. \tag{1.44}$$

*8. SPHERICAL TENSOR AND STRESS DEVIATOR

The octahedral normal stress may be interpreted as the mean normal stress at a given point in a body; using it, we construct the state of stress at a given point represented by the following stress tensor

$$\begin{pmatrix} N_0 & 0 & 0 \\ 0 & N_0 & 0 \\ 0 & 0 & N_0 \end{pmatrix}; \tag{1.45}$$

[1] Here and subsequently an asterisk indicates the chapters, sections and parts thereof which the reader may easily omit in his first acquaintance with the subject-matter without impairing continuity.

in this case the three principal stresses are the same, as are therefore the stresses over all the areas at the given point; Cauchy's stress surface and Lamé's stress ellipsoid appear to be spheres and the tensor (1.45) is thus called *spherical*. For the sake of convenience we introduce the concept of addition (subtraction) of tensors. The sum (difference) of two tensors is defined as a tensor whose components are equal to the sums (differences) of the respective components of these tensors.

Let us take an arbitrary stress tensor

$$(S)=\begin{pmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{pmatrix}. \qquad (1.46)$$

We add to it and then subtract from it the spherical tensor (1.45) where N_0 has the value (1.43):

$$(S)=\begin{pmatrix} X_x-N_0 & Y_x & Z_x \\ X_y & Y_y-N_0 & Z_y \\ X_z & Y_z & Z_z-N_0 \end{pmatrix}+\begin{pmatrix} N_0 & 0 & 0 \\ 0 & N_0 & 0 \\ 0 & 0 & N_0 \end{pmatrix}. \qquad (1.47)$$

The stress tensor can thus be represented, in general, as the sum of two tensors. The first of these tensors is called the *stress deviator*; briefly we denote it by (D_S).

The representation of the stress tensor in the form (1.47) is of cardinal importance in the investigation of the behaviour of elastic and plastic bodies under loading; but even now it is easy to perceive the importance of resolving the state of stress (1.45) into two component states (1.47). The spherical tensor (1.45) isolates from an arbitrary state of stress an all-round tension (compression), uniform in all directions, involving only the change of volume of the given element of a body but no change in its shape; therefore, the stress deviator (D_S) represents such a state of an element in which the shape of the element distorts without change of volume.

Consider the invariants of the stress deviator. Its first (linear) invariant is equal to zero, since an all-round tension-compression is excluded; indeed, by using formulas (1.33), we obtain

$$\Theta=(X_x-N_0)+(Y_y-N_0)+(Z_z-N_0)=$$

$$=X_x+Y_y+Z_z-3\frac{X_x+Y_y+Z_z}{3}=0. \qquad (1.48)$$

The second (quadratic) invariant is

$$\begin{aligned}\mathrm{H} &= (Y_y - N_0)(Z_z - N_0) + (X_x - N_0)(Z_z - N_0) + \\ &+ (X_x - N_0)(Y_y - N_0) - Y_z Z_y - Z_x X_z - Y_x X_y = Y_y Z_z + X_x Z_z + \\ &+ X_x Y_y - 2(X_x + Y_y + Z_z) N_0 + 3N_0^2 - Y_x^2 - Z_x^2 - X_y^2 = Y_y Z_z + \\ &+ X_x Z_z + X_x Y_y - 3N_0^2 - Y_z^2 - Z_x^2 - X_y^2 = \\ &= \frac{1}{3}\left[Y_y Z_z + X_x Z_z + X_x Y_y - X_x^2 - Y_y^2 - Z_z^2 - 3Y_z^2 - 3Z_x^2 - 3X_y^2\right] = \\ &= -\frac{1}{6}\left[(X_x - Y_y)^2 + (Y_y - Z_z)^2 + (Z_z - X_x)^2 + 6(X_y + Y_z + Z_x)^2\right].\end{aligned} \tag{1.49}$$

If we take the principal axes for co-ordinate axes, i.e., if we set

$$X_x = N_i; \quad Y_y = N_2; \quad Z_z = N_3; \quad X_y = Y_z = Z_x = 0,$$

this expression is simplified and made to assume the form

$$\mathrm{II} = -\frac{1}{6}\left[(N_1 - N_2)^2 + (N_2 - N_3)^2 + (N_3 - N_1)^2\right], \tag{1.50}$$

i.e., except for a numerical factor, it coincides with the square of the octahedral shearing stress (1.44).

The third (cubic) invariant is

$$\Lambda = \begin{vmatrix} X_x - N_0 & Y_x & Z_x \\ X_y & Y_y - N_0 & Z_y \\ X_z & Y_z & Z_z - N_0 \end{vmatrix}.$$

For the sake of simplicity, we take here too the principal axes for co-ordinate axes; then, writing out the determinant, we obtain

$$\Lambda (N_1 - N_0)(N_2 - N_0)(N_3 - N_0). \tag{1.51}$$

By direct calculation one can prove that this expression may be represented in the alternative form

$$\Lambda = \frac{1}{3}\left[(N_1 - N_0)^3 + (N_2 - N_0)^3 + (N_3 - N_0)^3\right]. \tag{1.52}$$

Then it is evident that

$$\sqrt[3]{\Lambda} = \sqrt[3]{\frac{1}{3}\left[(N_1 - N_0)^3 + (N_2 - N_0)^3 + (N_3 - N_0)^3\right]}$$

represents the average cubic deviation of a given state of stress (N_1, N_2, N_3) from the mean hydrostatic stress

$$N_0 = \frac{1}{3}(N_1 + N_2 + N_3). \tag{1.53}$$

We revert to the quadratic invariant (1.50) and rewrite it as

$$\mathrm{H} = -\frac{1}{3}(N_1^2 + N_2^2 + N_3^2 - N_1N_2 - N_2N_3 - N_3N_1). \quad (1.54)$$

By analogy with (1.52) we now form the square of the average quadratic deviation of a given state of stress (N_1, N_2, N_3) from the mean hydrostatic stress (1.53):

$$\Delta^2 = \frac{1}{3}[(N_1 - N_0)^2 + (N_2 - N_0)^2 + (N_3 - N_0)^2] = \\ = \frac{1}{3}(N_1^2 + N_2^2 + N_3^2 - 3N_0^2).$$

Substituting here the value of N_0 from (1.43), we find that

$$\Delta^2 = \frac{2}{9}(N_1^2 + N_2^2 + N_3^2 - N_1N_2 - N_2N_3 - N_3N_1),$$

and hence, comparing with (1.54),

$$\Delta^2 = -\frac{2}{3}\mathrm{H}. \quad (1.55)$$

Now comparing (1.48), (1.52) and (1.55) we come to the conclusion that the linear invariant of the stress deviator indicates the absence of compression-tension "on the average"; the quadratic and cubic invariants characterise, respectively, the average quadratic and cubic deviations of the state of stress N_1, N_2, N_3 from the mean hydrostatic stress corresponding to the stress tensor at a given point in a body [1].

9. GENERALISATION OF THE LAW OF RECIPROCITY OF STRESSES. EXAMPLES

To conclude this chapter let us generalise the law of reciprocity of shearing stresses (1.6).

Consider (Fig. 19) two areas, with normals x and v, which pass through a point M, and denote the total stresses over them by P_x and P_v, respectively. According to the notation here accepted

$$P_v \cos(P_v x) = X_v.$$

Furthermore,

$$X_x = P_x \cos(P_x x); \quad Y_x = P_x \cos(P_x y); \\ Z_x = P_x \cos(P_x z).$$

[1] The interpretation of the quadratic invariant as the average quadratic deviation was first given by V. V. Novozhilov (1952) and S. D. Ponomarev (1953).

Introducing this in the first of equations (1.8) and taking into account (1.6), we obtain

$$P_v \cos(P_v x) = X_v = P_x [\cos(vx)\cos(P_x x) + \\ + \cos(vy)\cos(P_x y) + \cos(vz)\cos(P_x z)] = P_x \cos(P_x v),$$

but once again, according to our notation,

$$P_x \cos(P_x v) = V_x;$$

thus,

$$X_v = V_x; \tag{1.56}$$

hence, the projections of the total stresses P_v and P_x over the areas concerned, respectively, on the normals x and v to these

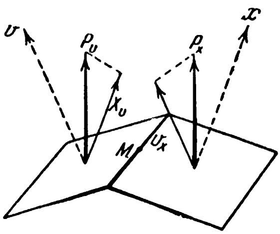

Fig. 19

areas are equal to each other (see Fig. 19). The law of reciprocity (1.6) now follows as a special case if the areas are mutually perpendicular.

Exercises to Sections 2-4.

1. In the wall of the dam represented in Fig. 6 there are induced the following stresses:

$$X_x = -\gamma y; \quad Y_y = \left(\frac{p}{\tan\beta} - \frac{2\gamma}{\tan^3\beta}\right)x + \left(\frac{\gamma}{\tan^2\beta} - p\right)y; \; Y_x = X_y = -\frac{\gamma}{\tan^2\beta}x;$$

$$Y_x = Z_y = X_z = Z_x = Z_z = 0.$$

Substitute these expressions in equations (1.5) and check in what conditions they will be satisfied. Find the load acting on the face Oy (i.e., for $x=0$); by using conditions (1.8) find also the load on the face OA (for $x = y\tan\beta$).

2. In the circular bar of diameter $2r$ (Fig. 7), there occur the following stresses:

$$X_x = Y_y = Z_z = X_y = Y_x = 0; \quad X_z = Z_x = -G\tau y; \quad Y_z = Z_y = G\tau x.$$

Check whether they satisfy equations (1.5) and in what conditions. Prove by the use of equations (1.8) that the lateral surface is free of load (i.e., $X_v = Y_v = Z_v = 0$).

Hint: for a point on the lateral surface

$$\cos(vx) = \frac{x}{r}; \quad \cos(vy) = \frac{y}{r}.$$

3. In the straight bar under pure bending (with the system of co-ordinate axes as shown in Fig. 8) there occur the stresses

$$X_x = -\frac{E_z}{\rho}; \quad Y_y = Z_z = X_y = Y_z = Z_x = 0.$$

Do the same as in the preceding exercise.

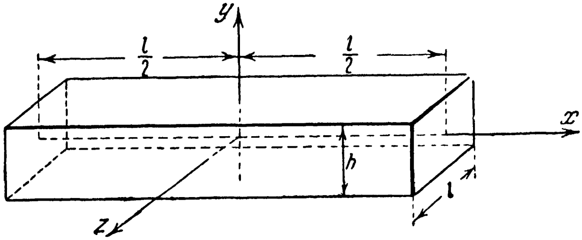

Fig. 20

4. In one of the cases of bendnig a rod of rectangular cross section the following stresses are set up (Fig. 20):

$$X_x = \frac{12}{h^3}\left(M_0 + \frac{ql^2}{8} - \frac{qh^2}{20}\right)y - \frac{6q}{h^3}x^2y + \frac{4q}{h^3}y^3;$$

$$Y_y = -\frac{6q}{h^3}\left(\frac{y^3}{3} - \frac{h^2}{4}y + \frac{h^3}{12}\right);$$

$$X_y = -\frac{6q}{h^3}\left(\frac{h^2}{4} - y^2\right)x;$$

$$Y_z = Z_x = Z_z = 0.$$

Check whether these stresses satisfy equations (1.5); find the loads acting on the faces of the rod.

II

Geometrical Theory of Strain

10. DISPLACEMENT COMPONENTS AND STRAIN COMPONENTS, AND RELATION BETWEEN THEM

Let us take an elastic body and fix it in such a way as to prevent its displacement as an absolutely rigid body. Then displacements of each of its points will be caused only by deformations.

Consider any point $M(x, y, z)$ in the body fixed as indicated above (Fig. 21). The point M will be displaced into a new position M' as a result of the deformation produced. We designate the projections of the displacement MM' on the co-ordinate axes by

$$u, \quad v, \quad w; \tag{2.1}$$

since displacements vary from point to point, *the projections of displacements* (2.1) *are functions of position*

$$\left.\begin{aligned} u &= f_1(x, y, z); \\ v &= f_2(x, y, z); \\ w &= f_3(x, y, z). \end{aligned}\right\} \tag{2.2}$$

Let us now pass from displacements to deformations. We isolate in the elastic body an infinitesimal parallelepiped (Fig. 22) with edges dx, dy, dz. During the deformation of the body it will displace and deform: the length of its edges will change and the initially right angles between the faces will distort.

To estimate the deformation of the elastic body at the given point M it is necessary to examine the elongations (linear deformations) of the edges dx, dy, dz of the isolated parallelepiped and the distortions of the angles *1M2*, *1M3*, *2M3* (shears or angular deformations). For this purpose consider the projections of the parallelepiped on the co-ordinate planes; obviously, the deformation of the parallelepiped itself can be deduced from the deformation of these three projections. In future we shall restrict our-

selves to very small deformations, in which case the subsequent treatment may be greatly simplified.

Take, for instance, the projection of the element $M123$ on the plane Oxy. Before deformation, the lengths of the edges are

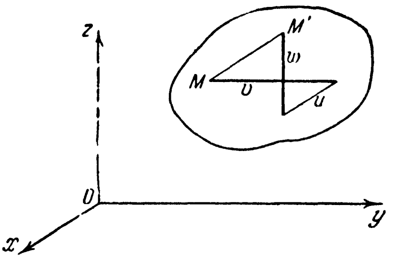

Fig. 21

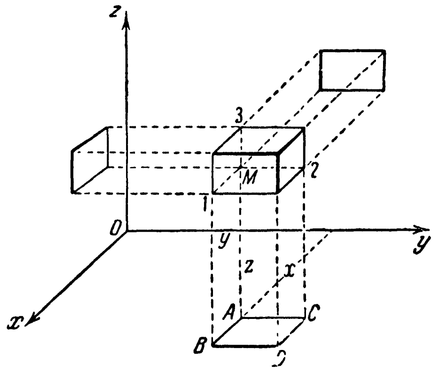

Fig. 22

(Fig. 23): $AB = dx$, $AC = dy$. After deformation, they occupy the position $A'B'$ and $A'C'$. We now fix attention on the projection of

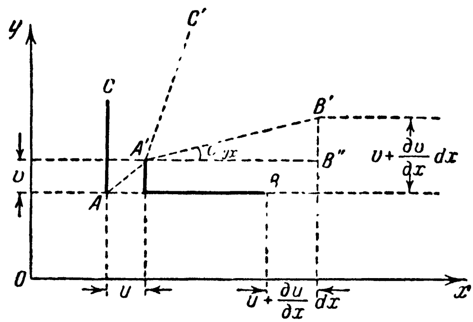

Fig. 23

AB. If the displacement of the point A along the axis Ox is u, the corresponding displacement of the point B is

$$u + \delta u = u + \frac{\partial u}{\partial x} dx,$$

where δu is the increment; since the point B differs from the point A only by the co-ordinate x, the small increment δu in the last formula is replaced, to infinitesimal quantities of the second order, by the partial differential of the function u with respect

to the variable x. Likewise, if the displacement of the point A along the axis Oy is v, the displacement of the point B along the same axis is expressed as

$$v+\frac{\partial v}{\partial x}dx.$$

The projection of the absolute elongation of the segment AB on the axis Ox is

$$\delta(dx)=\frac{\partial u}{\partial x}dx;$$

the unit elongation of this edge is

$$e_{xx}=\frac{\delta(dx)}{dx}=\frac{\partial u}{\partial x}.$$

By reasoning analogously, we obtain for the unit elongation of the edge AC directed along the axis Oy

$$e_{yy}=\frac{\partial v}{\partial y};$$

in the same way, for the edge parallel to the axis Oz (i.e., the edge $M3$ in Fig. 22) we have

$$e_{zz}=\frac{\partial w}{\partial z}.$$

We have obtained the formulas of linear deformations (elongations) at the given point M of the body in the direction of the three co-ordinate axes.

Let us now turn to the analysis of angular deformations. We easily find the angle of rotation α_{yx} of the edge $AB=dx$ in the plane Oxy:

$$\alpha_{yx}\approx\tan\alpha_{yx}=\frac{B'B''}{A'B''}=\frac{\frac{\partial v}{\partial x}dx}{dx+\frac{\partial u}{\partial x}dx}=\frac{\frac{\partial v}{\partial x}}{1+\frac{\partial u}{\partial x}}.$$

Since we have confined ourselves to the case of very small deformations, we may omit the quantity $\frac{\partial u}{\partial x}=e_{xx}$ in the denominator of the last expression as negligibly small compared with unity; we get

$$\alpha_{yx}=\frac{\partial v}{\partial x}. \qquad (2.3)$$

Similarly, we obtain the angle of rotation of the edge $AC=dy$ in the plane Oxy:

$$\alpha_{xy}\approx\tan\alpha_{xy}=\frac{\partial u}{\partial y}. \qquad (2.4)$$

We now can easily find the shearing strain, i.e., the distortion of the right angle BAC:

$$e_{xy} = \alpha_{yx} + \alpha_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \tag{2.5}$$

Formula (2.5) gives the expression of the shear angle occurring in the plane Oxy.

Similarly, we obtain the expressions of shearing strains in the other two co-ordinate planes by a cyclic change between the letters according to the scheme shown in Fig. 24.

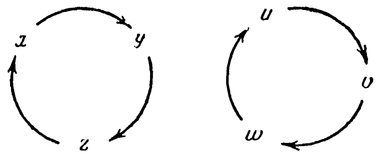

Fig. 24

Collecting together the above results, we get six basic relations characterising deformation:

(a) unit elongations (extensional strains or extensions)

$$e_{xx} = \frac{\partial u}{\partial x}; \quad e_{yy} = \frac{\partial v}{\partial y}; \quad e_{zz} = \frac{\partial w}{\partial z};$$

(b) shearing strains

$$\left.\begin{aligned} e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \\ e_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \\ e_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \end{aligned}\right\} \tag{2.6}$$

These equations were derived by A. L. Cauchy.

Sometimes the halves of the shear angles are introduced in formulas (2.6):

$$\bar{e}_{xy} = \frac{1}{2} e_{xy}; \quad \bar{e}_{yz} = \frac{1}{2} e_{yz}; \quad \bar{e}_{zx} = \frac{1}{2} e_{zx};$$

then the last three of them can be written down as

$$\left.\begin{aligned} 2\bar{e}_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \\ 2\bar{e}_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \\ 2\bar{e}_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \end{aligned}\right\} \tag{2.6a}$$

They are occasionally found to be more convenient in this form.

As regards the signs of the strains e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy} we should note the following. If the function u grows with increasing x, i.e., $e_{xx}=\frac{\partial u}{\partial x}>0$, we are obviously dealing with an increase in the length dx (the displacement of the point B to the right, according to Fig. 23, is greater than that of the point A). Thus, if $e_{xx}>0$, the result is elongation, and if $e_{xx}<0$, the result is contraction; the same rule holds true for e_{yy} and e_{zz}. Furthermore, if the function v grows with increasing x, then [formula (2.3)] $\alpha_{yx}=\frac{\partial v}{\partial x}>0$; in this case (Fig. 23) the segment AB rotates from the $+x$ axis to the $+y$ axis; similarly, for $\alpha_{xy}=\frac{\partial u}{\partial y}>0$ the segment AC rotates from the $+y$ axis to the $+x$ axis. Hence, it follows, according to (2.5), that the shearing strain $e_{xy}>0$, if the right angle between the $+x$ and $+y$ axes decreases. The same rule applies to the shearing strains in the other two planes. Summarising what has been said above, we obtain the following sign rule for deformations:

(a) *elongations* along the co-ordinate axes *correspond to positive linear deformations* e_{xx}, e_{yy}, e_{zz}, and contractions to negative;

(b) *a decrease in the angles* between the positive directions of the axes *corresponds to positive angular deformations* (shear deformations) e_{yz}, e_{zx}, e_{xy}, and an increase in the same angles to negative.

Formulas (2.6) show that six functions e_{xx}, e_{yy}, e_{zz}, e_{yx}, e_{zx}, e_{xy}, which are called strain components or components of strain, are linearly expressed in terms of nine partial derivatives of the displacement components u, v, w:

$$\left.\begin{array}{lll}\frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z}; \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z}; \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z}.\end{array}\right\} \qquad (2.7)$$

The components of the principal diagonal of this matrix are elongations [see the first three relations of (2.6)]. The remaining components represent the angles of rotation of the edges of the elementary parallelepiped *M123*, shown in Fig. 22, about the x, y, z axes; this can be seen, for instance, from formulas (2.3) and (2.4); in the notation accepted therein matrix (2.7) is written as

$$\left.\begin{array}{lll}e_{xx}, & \alpha_{yx}, & \alpha_{zx}; \\ \alpha_{xy}, & e_{yy}, & \alpha_{zy}; \\ \alpha_{xz}, & \alpha_{yz}, & e_{zz}.\end{array}\right\} \qquad (2.8)$$

These notations are found to be convenient; the second subscript of the letter α indicates the elementary segment (dx, dy, dz) the rotation of which is being considered; the first subscript denotes the axis in the direction of which the rotation takes place. For instance, $\alpha_{zy}=\frac{\partial w}{\partial y}$ designates the angle of rotation of the element dy in the direction from the y axis to the z axis.

We should note the important fact that it is impossible to set up a system of equations inverse to equations (2.6), i.e., to express nine components of matrix (2.7) in terms of six components of strain (2.6); equations (2.6) are inadequate. This

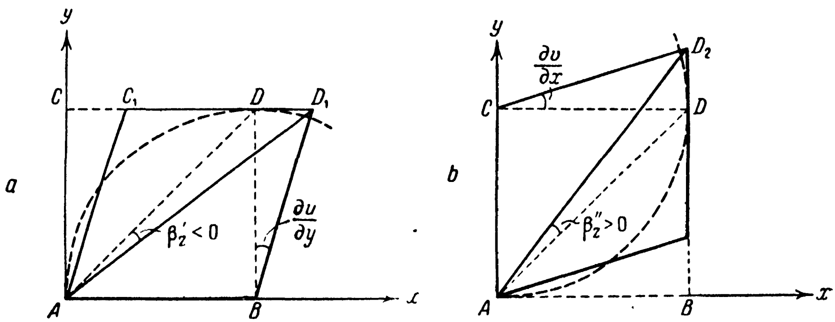

Fig. 25

is due to the fact that our geometrical representation of deformations at a given point is as yet incomplete; to complete it and to obtain symmetry in calculations let us introduce three more components; let the element *M123* in Fig. 22 be a cube ($dx=dy=$ $=dz$) and consider the angles of rotation of its diagonals about the x, y, z axes in the case when elongations $e_{xx}=e_{yy}=e_{zz}=0$; Fig. 25 *a* shows the projection *ABCD* of the cube under consideration on the plane *Axy*.

Obviously, the angle of rotation of the diagonal of the cube about the z axis is equal to the angle of rotation of the projection *AD* of this diagonal about the point *A*. First consider the particular case of "simple shear" parallel to the x axis (Fig. 25 *a*); for the angle of rotation β'_z of the diagonal *AD* we get then

$$\beta_z'=\angle DAD_1\approx\frac{1}{2}\angle DBD_1.$$

This can be proved if we construct a circle ADD_1 with centre at B and assume, disregarding small quantities of the second order, that the point D_1 lies on the circumference; the angles DAD_1 and DBD_1 are then subtended by the common arc DD_1; hence

there follows the foregoing approximate equality which implies that

$$\beta_z' = -\frac{1}{2}\frac{\partial u}{\partial y}.$$

The minus sign is taken because in the case under consideration

$$\frac{\partial u}{\partial y} > 0, \text{ and } \beta_z' < 0\text{[1]}.$$

In the case of simple shear along the y axis (Fig. 25 b) we find in a similar way

$$\beta_z'' = \frac{1}{2}\frac{\partial v}{\partial x}.$$

In the general case of shear we have

$$\beta_z = \beta_z' + \beta_z'' = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(\alpha_{xy} - \alpha_{yx}).$$

Reasoning in the same way with regard to the yz and zx planes, we obtain analogous formulas for the angles of rotation of the diagonal of the elementary cube about the x and y axes. It is usual practice to introduce double angles of rotation; they are denoted by 2ω with appropriate subscripts. Thus, we get

$$\left.\begin{aligned} 2\omega_x &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \\ 2\omega_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \\ 2\omega_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \end{aligned}\right\} \qquad (2.9)$$

The quantities ω_x, ω_y, ω_z are called components of rotation; to complete the picture of deformations at a given point equations (2.9) should be added to equations (2.6). Let us examine the deformation in a simple but very important case when displacements u, v, w are linear functions of the co-ordinates of a point, namely

$$\left.\begin{aligned} u &= u_0 + c_{11}x + c_{12}y + c_{13}z; \\ v &= v_0 + c_{21}x + c_{22}y + c_{23}z; \\ w &= w_0 + c_{31}x + c_{32}y + c_{33}z. \end{aligned}\right\} \qquad (2.10)$$

---

[1] We assume, as is customary in kinematics, that counterclockwise rotation about an axis is positive.

The components of strain (2.6) and the components of rotation (2.9) will then be constants; such a deformation is defined as homogeneous. During the deformation a given body is converted into another body; each point (x, y, z) moves to another point:

$$x_1 = x + y;$$
$$y_1 = y + v;$$
$$z_1 = z + w.$$

It is evident that in the case of homogeneous deformation (2.10) each plane (or straight line) is transformed into a plane (or a straight line): two parallel planes (or straight lines) are likewise transformed into two parallel planes (or straight lines); a rectangular parallelepiped is transformed, generally speaking, into an oblique parallelepiped. Inasmuch as in deriving equations (2.6) we have disregarded small quantities of higher order, the deformation defined by these equations will be homogeneous, generally speaking, only in a very small region isolated in the body; hence, the elementary parallelepiped *M123* (Fig. 22) will be converted into an oblique parallelepiped in the general case as well; its opposite faces will remain plane and mutually parallel.

Consider also equations (2.10) in the case when all strain components (2.6) are equal to zero and the body does not deform; we easily find then

$$c_{11} = c_{22} = c_{33} = 0; \quad c_{21} = -c_{12}; \quad c_{32} = -c_{23}; \quad c_{31} = -c_{12}.$$

Introducing more compact notations

$$c_{32} = p; \quad c_{13} = q; \quad c_{21} = r,$$

we write down (2.10) as

$$\left.\begin{aligned} u &= u_0 + qz - ry; \\ v &= v_0 + rx - pz; \\ w &= w_0 + py - qx. \end{aligned}\right\} \qquad (2.11)$$

These formulas coincide with the formulas of kinematics for infinitesimal displacements of an absolutely rigid body; u_0, v_0, w_0 are the translatory displacements of the body; p, q, r are the angles of rotation about the co-ordinate axes. Furthermore, we have

$$2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = c_{32} - c_{23} = 2c_{32} = 2p; \quad \omega_x = p;$$

similarly

$$\omega_y = q; \quad \omega_z = r; \qquad (2.11a)$$

hence, the components ω_x, ω_y, ω_z introduced here represent the angles of rotation of the given element about the axes.

In the theory of elasticity it is the custom to say that formulas (2.11) express a "rigid body displacement".

Now consider the inverse case when the components of strain (2.6) are different from zero but the components of rotation (2.9) vanish; then

$$\frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}; \quad \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}; \quad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y},$$

but this is the condition that the expression

$$u\,dx+v\,dy+w\,dz$$

is the total differential of a certain function

$$\Phi(x, y, z),$$

i. e.,

$$u=\frac{\partial \Phi}{\partial x}, \quad v=\frac{\partial \Phi}{\partial y}, \quad w=\frac{\partial \Phi}{\partial z}.$$

It is said in such a case that there occurs "pure deformation" and that displacements have a potential $\Phi(x, y, z)$.

11. COMPATIBILITY EQUATIONS

Summarising the foregoing conclusions we see that the displacement of a given point in an elastic body is determined by three functions (see 2.2):

$$u, \quad v, \quad w;$$

the deformation at the given point is determined by six functions

$$e_{xx}, \quad e_{yy}, \quad e_{zz}, \quad e_{yz}, \quad e_{zx}, \quad e_{xy}. \tag{2.12}$$

However, equations (2.6) show that if three functions (2.2) are specified, all six components of strain (2.12) will be determined thereby, being expressible in terms of the first derivatives of functions (2.2); hence, it may be foreseen that six strain components (2.12) cannot be prescribed arbitrarily; there must exist certain relations among them to the derivation of which we now proceed. These relations number six, and they fall into two groups.

First group. Let us differentiate the first two of equations (2.6) as follows:

$$\frac{\partial^2 e_{xx}}{\partial y^2}=\frac{\partial^3 u}{\partial x\,\partial y^2}; \quad \frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^3 v}{\partial y\,\partial x^2};$$

adding up these equations by members, we get

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\frac{\partial^2 e_{xy}}{\partial x\,\partial y}.$$

This is one of the required relations among strains. By a cyclic change between its letters, we get the other two equalities. This leads us to the first group of relations

$$\left.\begin{aligned}
\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}&=\frac{\partial^2 e_{xy}}{\partial x\,\partial y};\\
\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}&=\frac{\partial^2 e_{yz}}{\partial y\,\partial z};\\
\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}&=\frac{\partial^2 e_{zx}}{\partial z\,\partial x}.
\end{aligned}\right\}\tag{2.13}$$

Second group. We differentiate the last three of equations (2.6) as follows:

$$\begin{aligned}
\frac{\partial e_{yz}}{\partial x}&=\frac{\partial^2 w}{\partial y\,\partial x}+\frac{\partial^2 v}{\partial z\,\partial x}\quad\Big|\;+\\
\frac{\partial e_{zx}}{\partial y}&=\frac{\partial^2 u}{\partial z\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial y}\quad\Big|\;+\\
\frac{\partial e_{xy}}{\partial z}&=\frac{\partial^2 v}{\partial x\,\partial z}+\frac{\partial^2 u}{\partial y\,\partial z}\quad\Big|\;-
\end{aligned}$$

We change the signs in the last of the resulting equations and then add them all up by members:

$$\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}=2\frac{\partial^2 w}{\partial x\,\partial y}.$$

Differentiating this equation once more with respect to z and observing that

$$\frac{\partial^3 w}{\partial x\,\partial y\,\partial z}=\frac{\partial^2 e_{zz}}{\partial x\,\partial y},$$

we get

$$\frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}\right)=2\frac{\partial^2 e_{zz}}{\partial x\,\partial y}.\tag{2.14}$$

This is one of the required relations of the second group. By a cyclic change between the letters in equation (2.14), we obtain two more equations of the same form. Adding these relations to

relations (2.13), we obtain the following system of equations:

$$\left.\begin{aligned}
&\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2 e_{xy}}{\partial x\,\partial y};\\
&\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}=\frac{\partial^2 e_{yz}}{\partial y\,\partial z};\\
&\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}=\frac{\partial^2 e_{zx}}{\partial z\,\partial x};\\
&\frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}\right)=2\,\frac{\partial^2 e_{zz}}{\partial x\,\partial y};\\
&\frac{\partial}{\partial x}\left(\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}\right)=2\,\frac{\partial^2 e_{xx}}{\partial y\,\partial z};\\
&\frac{\partial}{\partial y}\left(\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}\right)=2\,\frac{\partial^2 e_{yy}}{\partial z\,\partial x}.
\end{aligned}\right\}\quad(2.15)$$

We have justified the necessity of the existence of these relations analytically, by comparing the number of functions (2.2) to the number of functions (2.12) and taking into account the relations between them (2.6); this same purpose can be achieved by a geometrical approach. To this end, let us imagine an elastic body cut into small parallelepipeds and assign each of them the deformation defined by six quantities (2.12); it is easy to conceive that if the components of strain (2.12) are not connected by certain relations, it is impossible to make a continuous (deformed) solid from individual deformed parallelepipeds. Equations (2.15) furnish us with these relations; if we do not take them into consideration in specifying strains (2.12), there will occur infinitesimal discontinuities at each point of the body after deformation. These considerations lend a new meaning to equations (2.15), and from this new point of view equations (2.15) are called the equations of continuity or compatibility of strain components. They were derived by Saint-Venant and are often called Saint-Venant's equations. We shall make another important observation.

If, given the loads acting on a body, we are in a position to find directly the displacements of its points u, v, w, strains (2.12) can be calculated thereupon according to equations (2.6); the compatibility conditions will then be satisfied automatically, since they have been derived from equations (2.6) and follow from them as a consequence [1].

If, however, given the loads acting on a body, we find stresses and then strains (2.12), it will be necessary to satisfy simulta-

[1] From this point of view equations (2.15) are sometimes called Saint-Venant's identities.

neously the equations of compatibility (2.15) as well; otherwise deformations will be incompatible and we shall not be able to find displacements (2.2) from equations (2.6) as the latter will involve intrinsic contradictions.

Consider this question in greater detail. By introducing the components of rotation we find it possible to write down a system of equations inverse to (2.6), i.e., to solve a set of simultaneous equations (2.6) and (2.9) with respect to derivatives (2.7) and to express them in terms of components of strain and rotation:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}=e_{xx};\quad \frac{\partial u}{\partial y}=(\bar{e}_{xy}-\omega_z);\quad \frac{\partial u}{\partial z}=(\bar{e}_{xz}+\omega_y);\\
&\frac{\partial v}{\partial x}=(\bar{e}_{xy}+\omega_z);\quad \frac{\partial v}{\partial y}=e_{yy};\quad \frac{\partial v}{\partial z}=(\bar{e}_{yz}-\omega_x);\\
&\frac{\partial w}{\partial x}=(\bar{e}_{xz}-\omega_y);\quad \frac{\partial w}{\partial y}=(\bar{e}_{yz}+\omega_x);\quad \frac{\partial w}{\partial z}=e_{zz}.
\end{aligned}\right\} \tag{2.16}$$

The matrix of components (2.7) can now be represented conventionally as the sum of two matrices, so that each component of matrix (2.7) is equal to the sum of the respective components of the added up matrices:

$$\begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{vmatrix}
=
\begin{vmatrix}
e_{xx} & \bar{e}_{xy} & \bar{e}_{xz}\\
\bar{e}_{xy} & e_{yy} & \bar{e}_{yz}\\
\bar{e}_{xz} & \bar{e}_{yz} & e_{zz}
\end{vmatrix}
+
\begin{vmatrix}
0 & -\omega_z & \omega_y\\
\omega_z & 0 & -\omega_x\\
-\omega_y & \omega_x & 0
\end{vmatrix}. \tag{2.17}$$

The first matrix in the right-hand member is symmetric: it determines a pure deformation (without rotation). The second matrix is antisymmetric: we see that it determines a rigid body rotation (without deformation). Our treatment can be correlated with the theory of tensors [1], thus making it possible to formulate the last result as follows: the tensor of small deformation (2.7) can be resolved into a symmetric tensor of pure deformation and an antisymmetric tensor of rigid body rotation. Tensor (2.7) is sometimes called the *tensor of relative displacements*. In fact, consider an infinitesimal parallelepiped with edges $dx=1$, $dy=1$, $dz=1$; it is obvious that

$$\frac{\partial u}{\partial x}=e_{xx};\quad \frac{\partial v}{\partial y}=e_{yy};\quad \frac{\partial w}{\partial z}=e_{zz}$$

represent the displacements u, v, w of the vertices of this parallelepiped (along the co-ordinate axes) relative to one of them.

[1] See the above-cited N. I. Muskhelishvili's book.

The quantities

$$\frac{\partial v}{\partial x}=\alpha_{yx}, \quad \frac{\partial u}{\partial y}=\alpha_{xy}, \ \ldots$$

are the angles of rotation of its edges with respect to their original positions.

Equalities (2.16) may be regarded as differential equations in u, v, w. If the components of strain and rotation are given at all points of a body, the displacements themselves of the points of the body u, v, w can be found by integrating equations (2.16). Carrying out this integration, we shall, in effect, convince ourselves of the necessity of satisfying the compatibility conditions (2.6).

Let us turn to the equations in the first line of (2.16) from which we shall seek the function $u(x, y, z)$; we note that, according to the meaning of these equations, their right-hand members must be partial derivatives of one and the same function; hence, the following three conditions should be satisfied:

$$\frac{\partial e_{xx}}{\partial y}=\frac{\partial}{\partial x}(\bar{e}_{xy}-\omega_z); \quad \frac{\partial e_{xx}}{\partial z}=\frac{\partial}{\partial x}(\bar{e}_{xz}+\omega_y); \qquad (2.18)$$

$$\frac{\partial}{\partial y}(\bar{e}_{xz}+\omega_y)=\frac{\partial}{\partial z}(\bar{e}_{xy}-\omega_z),$$

which result from eliminating the function u from equations (2.16). Using (2.18), we find

$$\frac{\partial \omega_z}{\partial x}=\frac{\partial \bar{e}_{xy}}{\partial x}-\frac{\partial e_{xx}}{\partial y};$$

$$\frac{\partial \omega_y}{\partial x}=-\frac{\partial \bar{e}_{xz}}{\partial x}+\frac{\partial e_{xx}}{\partial z};$$

$$\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}=\frac{\partial \bar{e}_{xy}}{\partial z}-\frac{\partial \bar{e}_{xz}}{\partial y}.$$

The last equation can be simplified on the basis of the following identity, which is obtained by differentiating equalities (2.9):

$$\frac{\partial \omega_x}{\partial x}+\frac{\partial \omega_y}{\partial y}+\frac{\partial \omega_z}{\partial z}=0, \qquad (2.19)$$

and we have then

$$\left.\begin{aligned}
&\frac{\partial \omega_z}{\partial x}=\frac{\partial \bar{e}_{xy}}{\partial x}-\frac{\partial e_{xx}}{\partial y};\\
&\frac{\partial \omega_y}{\partial x}=-\frac{\partial \bar{e}_{xz}}{\partial x}+\frac{\partial e_{xx}}{\partial z};\\
&\frac{\partial \omega_x}{\partial x}=\frac{\partial \bar{e}_{xz}}{\partial y}-\frac{\partial \bar{e}_{xy}}{\partial z}.
\end{aligned}\right\} \qquad (2.20)$$

By performing identical operations upon the second and third lines of equations (2.16), we eliminate v and w from them and obtain six more equations in addition to (2.20); in this way the nine partial derivatives of the components of rotation will be expressed in terms of partial derivatives of the components of strain; in other words, we obtain a system of equations for ω_x, ω_y, ω_z analogous to equations (2.16) for u, v, w. Hence, it is permissible to apply the foregoing procedure to the equations thus obtained and to eliminate ω_x, ω_y, ω_z from them; we shall again obtain nine equalities but they will contain only the second derivatives of the strain components. Only six of these equalities are different and they will coincide with equations (2.15). Thus, Saint-Venant's compatibility equations (2.15) are, in effect, the necessary conditions for displacements to be determined from (2.6) according to the given strain components.

Saint-Venant's conditions are simultaneously sufficient for this purpose, if we consider a simply connected body that has no through cavities. In the case of a multiply connected body, Saint-Venant's conditions also permit the determination of displacements u, v, w by integrating Cauchy's equations (2.6); these displacements, however, may now be represented by multiple-valued functions of x, y, z; besides Saint-Venant's conditions, it is necessary to introduce some additional conditions to make displacements single-valued, as it is required by the physical meaning of the problem.

The case of deformation of a multiply connected body is treated in detail in Chapter I and Appendix II of the above-cited N. I. Muskhelishvili's book.

*12. TENSOR CHARACTER OF THE STRAIN AT A GIVEN POINT IN A BODY

In Chapter I we introduced the concept of the stress tensor; from the viewpoint of the general theory of tensors the formal and at the same time basic feature of the tensor character of a state of stress at a given point is that, as one passes from the coordinate areas to an arbitrary area with outward normal v, the stress components X_v, Y_v, Z_v are expressed by formulas (1.8a), which are linear with respect to the original components (1.16) as well as to the direction cosines l, m, n. Under a complete transformation involving the passage from the axes x, y, z to new axes u, v, w, the components of the state of stress are expressed in terms of the original components according to formulas of the form of (1.12) and (1.13), which are linear with respect to the

original components (1.16) and quadratic (or so-called bilinear) with respect to the direction cosines of the new system (1.10).

In this section we shall show that the deformation of a body at a given point, determined by nine components of matrix (2.7), is also a tensor from the above point of view.

The deformation at a given point in a body will be completely determined if we can calculate the unit elongation of any infinitesimal segment drawn from the given point. Therefore, consider (Fig. 26) such a segment $AB=\rho$. We designate its projections on the co-ordinate axes by ξ, η, ζ. The drawing shows the co-ordinates of the initial point A of the segment and those of the terminal point B. In the process of deformation these points will

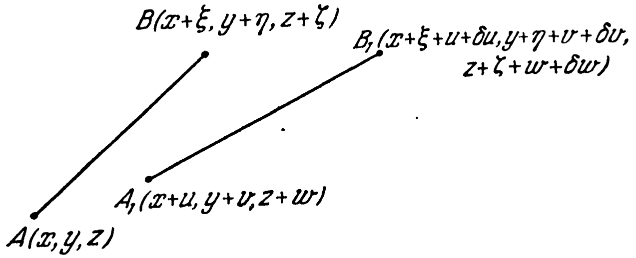

Fig. 26

occupy new positions A_1 and B_1; their co-ordinates are also shown in Fig. 26. The projections of the deformed segment A_1B_1 on the co-ordinate axes will obviously be obtained by subtracting the original co-ordinates from the new ones; for instance,

$$\xi_1 = x+\xi+u+\delta u-(x+u)=\xi+\delta u; \tag{2.21a}$$

in the same way we find

$$\eta_1=\eta+\delta v; \quad \zeta_1=\zeta+\delta w. \tag{2.21b}$$

Obviously, the quantities

$$\xi_1-\xi=\delta\xi; \quad \eta_1-\eta=\delta\eta; \quad \zeta_1-\zeta=\delta\zeta \tag{2.22}$$

represent the increments of the projections of the segment ρ in the process of deformation of the body while the quantities δu, δv, δw are the increments of the functions u, v, w as the point A moves to the point B, i.e., as the independent variables x, y, z are changed by the amounts

$$dx=\xi; \quad dy=\eta; \quad dz=\zeta.$$

These quantities are infinitely small because of the infinitesimal magnitude of the segment $\overline{AB}=\rho$; therefore, the infinitesimal increments δu, δv, δw of the functions u, v, w may be replaced by their differentials (omitting small quantities of higher order):

$$\delta u=\frac{\partial u}{\partial x}\xi+\frac{\partial u}{\partial y}\eta+\frac{\partial u}{\partial z}\zeta;$$

$$\delta v=\frac{\partial v}{\partial x}\xi+\frac{\partial v}{\partial y}\eta+\frac{\partial v}{\partial z}\zeta;$$

$$\delta w=\frac{\partial w}{\partial x}\xi+\frac{\partial w}{\partial y}\eta+\frac{\partial w}{\partial z}\zeta;$$

formulas (2.21a, b) then become

$$\left.\begin{aligned}\delta\xi&=\frac{\partial u}{\partial x}\xi+\frac{\partial u}{\partial y}\eta+\frac{\partial u}{\partial z}\zeta;\\ \delta\eta&=\frac{\partial v}{\partial x}\xi+\frac{\partial v}{\partial y}\eta+\frac{\partial v}{\partial z}\zeta;\\ \delta\zeta&=\frac{\partial w}{\partial x}\xi+\frac{\partial w}{\partial y}\eta+\frac{\partial w}{\partial z}\zeta.\end{aligned}\right\}\tag{2.23}$$

We divide these equalities by the original length of the segment ρ and introduce the notations

$$\left.\begin{aligned}&\frac{\xi}{\rho}=l; &&\frac{\eta}{\rho}=m; &&\frac{\zeta}{\rho}=n;\\ &\frac{\delta\xi}{\rho}=e_{x\rho}; &&\frac{\delta\eta}{\rho}=e_{y\rho}; &&\frac{\delta\zeta}{\rho}=e_{z\rho}.\end{aligned}\right\}\tag{2.24}$$

Here l, m, n are the direction cosines of the initial segment AB; $e_{x\rho}$, $e_{y\rho}$, $e_{z\rho}$ are the elongations of the projections of this segment per unit of its length. Formulas (2.23) then become

$$\left.\begin{aligned}e_{x\rho}&=\frac{\partial u}{\partial x}l+\frac{\partial u}{\partial y}m+\frac{\partial u}{\partial z}n;\\ e_{y\rho}&=\frac{\partial v}{\partial x}l+\frac{\partial v}{\partial y}m+\frac{\partial v}{\partial z}n;\\ e_{z\rho}&=\frac{\partial w}{\partial x}l+\frac{\partial w}{\partial y}m+\frac{\partial w}{\partial z}n.\end{aligned}\right\}\tag{2.25}$$

These relations are analogous in structure to relations (1.8a), in which the quantities

$$X_x,\quad Y_y,\quad Z_z,\quad Y_z,\quad Z_x,\quad X_y$$

represent the components of the stress tensor; hence, the components of matrix (2.7) entering in relations (2.25) are the components of a tensor which we have called the tensor of relative displacements; however, unlike (1.16), this tensor is asymmetric

since, in general,

$$\frac{\partial v}{\partial x} \neq \frac{\partial u}{\partial y}; \quad \frac{\partial w}{\partial y} \neq \frac{\partial v}{\partial z}; \quad \frac{\partial u}{\partial z} \neq \frac{\partial w}{\partial x}.$$

These inequalities are transformed into equalities only in the absence of rotation at a given point in a body ($\omega_x=\omega_y=\omega_z=0$), i.e., in the case of pure deformation (see end of Sec. 10).

The left-hand members of equalities (1.8a) and (2.25) are also identical in meaning; they are projections on the co-ordinate axes: in the first case — the projections of the total stress on the area with outward normal $v(l, m, n)$; in the second case — the quantities proportional to the elongations of the projections of a segment having the direction cosines l, m, n.

Now consider the elongation of the segment $AB=\rho$ itself; since

$$\rho^2=\xi^2+\eta^2+\zeta^2,$$

the elongation, to small quantities of the second order, will be obtained from the equality

$$\rho\delta\rho=\xi\delta\xi+\eta\delta\eta+\zeta\delta\zeta.$$

By dividing both members of this equality by ρ^2, we get

$$\frac{\delta\rho}{\rho}=\frac{\xi}{\rho}\frac{\delta\xi}{\rho}+\frac{\eta}{\rho}\frac{\delta\eta}{\rho}+\frac{\zeta}{\rho}\frac{\delta\zeta}{\rho}.$$

The left-hand member represents the unit elongation of the segment ρ:

$$\frac{\delta\rho}{\rho}=e_\rho;$$

by using notations (2.24) and equalities (2.25), we find

$$e_\rho=\frac{\partial u}{\partial x}l^2+\frac{\partial v}{\partial y}m^2+\frac{\partial w}{\partial z}n^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)lm+$$

$$+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)mn+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)nl \qquad (2.26)$$

or on the basis of Cauchy's equations (2.6)

$$e_\rho=e_{xx}l^2+e_{yy}m^2+e_{zz}n^2+2\bar{e}_{xy}lm+2\bar{e}_{yz}mn+2\bar{e}_{zx}nl. \qquad (2.27)$$

Equalities (2.26) and (2.27) confirm definitely the tensor character of the quantities involved in the matrices

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e_{xx} & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{yx} & \bar{e}_{yy} & \bar{e}_{yz} \\ \bar{e}_{zx} & \bar{e}_{zy} & \bar{e}_{zz} \end{pmatrix} \qquad (2.28)$$

for, knowing the components of these matrices, we can find the unit elongation of an arbitrary segment ρ drawn from a given point; this, obviously, determines completely the deformation of a body at a given point in the same way as the knowledge of nine components of the stress tensor (1.16) fully determines the state of stress at a given point in a body, i.e., allows the stresses to be determined over any area passing through this point. We have introduced the halves of the shear angles $\bar{e}_{xy}$, $\bar{e}_{yz}$, $\bar{e}_{zx}$ in relation* (2.27) and in the second of matrices (2.28) (cf. 2.6a); relation (2.27) has then assumed a form quite similar to (1.19). The first of matrices (2.28) was called the tensor of relative displacements; it can now be established that the second matrix defines the *strain tensor* at a given point. Relation (1.19) permitted us to give a geometrical representation of the stress distribution at a given point with the aid of Cauchy's stress surface; relation (2.27) makes it possible to give a similar picture to describe the deformation of a body at a given point. Indeed, we multiply both members of equality (2.27) by ρ^2 and take into consideration the first line of notation (2.24); it can then be rewritten as

$$\rho^2 e_\rho = e_{xx}\xi^2 + e_{yy}m^2 + e_{zz}n^2 + 2\bar{e}_{xy}\xi\eta + 2\bar{e}_{yz}\eta\zeta + 2\bar{e}_{zx}\zeta\xi. \tag{2.27'}$$

We take the point under investigation to be the origin of co-ordinates (ξ, η, ζ) and construct at it a vector equal in magnitude to the segment $\boldsymbol{\rho}$; then ξ, η, ζ will be the co-ordinates of the end of this vector; we shall adjust the length of the vector so as to have at all times

$$\rho^2 e_\rho = \pm b^2, \qquad \text{i.e.,} \qquad \rho = \sqrt{\frac{\pm b^2}{l_\rho}}. \tag{2.29}$$

Then (2.27') will be transformed into the equation of a surface of the second order

$$F(\xi, \eta, \zeta) = \pm b^2, \tag{2.30}$$

where

$$F(\xi, \eta, \zeta) = e_{xx}\xi^2 + e_{yy}\eta^2 + e_{zz}\zeta^2 + 2\bar{e}_{xy}\xi\eta + 2\bar{e}_{yz}\eta\zeta + 2\bar{e}_{zx}\zeta\xi. \tag{2.31}$$

Surface (2.30) is quite analogous to Cauchy's stress surface (1.23); it possesses the same properties and is called the *strain surface*. It is a central surface of the second order with centre at the point under investigation and it may be either an ellipsoid or the combination of a hyperboloid of one sheet and a hyperboloid of two sheets with the common asymptotic cone. If we draw radius-vectors $\boldsymbol{\rho}$ from its centre up to the intersection with the surface, we shall have from (2.29)

$$e_\rho = \pm \frac{b^2}{\rho^2}.$$

It follows that the unit elongation at the point in question measured in the direction of the vector ρ is inversely proportional to the square of the length of this vector. The whole series of such vectors characterises completely the deformation of the body at a given point and may, therefore, be regarded as a geometrical representation of the strain tensor (just as a straight line segment of given length and direction is a geometrical representation of a vector).

Surface (2.30), like stress surface (1.23), has three mutually orthogonal principal axes; if it is referred to these axes, the terms containing the products of co-ordinates disappear in equation (2.30), i.e., their coefficients become zero:

$$\bar{e}_{xy} = \bar{e}_{yz} = \bar{e}_{zx} = 0;$$

hence, shear deformations are absent in the principal planes. To determine the principal axes one must, according to the rule of analytic geometry, equate to zero the determinant

$$\begin{vmatrix} e_{xx} - e & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} - e & \bar{e}_{yz} \\ \bar{e}_{xz} & \bar{e}_{yz} & e_{zz} - e \end{vmatrix} = 0,$$

which gives for the determination of e the cubic equation

$$e^3 - \theta e^2 + \eta e - \lambda' = 0$$

similar to equation (1.32); its coefficients

$$\theta = e_{xx} + e_{yy} + e_{zz}, \tag{2.32a}$$

$$\eta = \begin{vmatrix} e_{xx} & \bar{e}_{xy} \\ \bar{e}_{xy} & e_{yy} \end{vmatrix} + \begin{vmatrix} e_{xx} & \bar{e}_{xz} \\ \bar{e}_{xz} & e_{zz} \end{vmatrix} + \begin{vmatrix} e_{yy} & \bar{e}_{yz} \\ \bar{e}_{yz} & e_{zz} \end{vmatrix}, \tag{2.32b}$$

$$\lambda' = \begin{vmatrix} e_{xx} & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} & \bar{e}_{yz} \\ \bar{e}_{xz} & \bar{e}_{yz} & e_{zz} \end{vmatrix}, \tag{2.32c}$$

represent invariants under a transformation of co-ordinates. Its three roots are real and represent *principal elongations* (elongations along the principal axes); denoting them by e_1, e_2, e_3, we can write down the equation of strain surface (2.30) in the principal axes

$$e_1\xi^2 + e_2\eta^2 + e_3\zeta^2 = \pm b^2. \tag{2.33}$$

The form of this surface, just as that of the stress surface, depends on the signs of the principal elongations e_1, e_2, e_3. If all three

elongations are of the same sign, the surface is an ellipsoid; in this case, in all directions at a given point there occurs either extension (if the principal elongations are positive) or compression (if the principal elongations are negative). If, however, the principal elongations are of different signs, surface (2.33) should be represented as a combination of hyperboloids of one and two sheets, with the asymptotic cone separating them. Let us construct a vector in the desired direction from the given point of the body (i.e., from the centre of the surface); if the vector intersects the hyperboloid of one sheet, extension occurs in this direction; in the directions which intersect the hyperboloid of two sheets, compression occurs; in the direction of the generators of the asymptotic cone the length of the vector becomes infinite; in these directions elongations are equal to zero.

*13. DILATATIONAL STRAIN. INVARIANTS OF THE STRAIN TENSOR

In addition to strain components (2.12), let us also find the dilatational strain. Consider an infinitesimal element of a body of volume $d\tau = dx\,dy\,dz$. It is easy to conceive that if shear deformations alone are produced in the given element without elongation of its edges, the unit change of volume of the element will be a small quantity of higher order compared to the shear deformations. Consequently, we may assume, neglecting small quantities of higher order, that the change of volume $d\tau$ will depend only on the elongations of the edges dx, dy, dz.

The new volume of the element after deformation will be

$$d\tau + \delta(d\tau) = dx(1 + e_{xx})\,dy(1 + e_{yy})\,dz(1 + e_{zz}).$$

Opening the parentheses in the right-hand member we get

$$d\tau + \delta(d\tau) = dx\,dy\,dz(1 + e_{xx} + e_{yy} + e_{zz} + \\ + e_{xx}e_{yy} + e_{xx}e_{zz} + e_{yy}e_{zz} + e_{xx}e_{yy}e_{zz}).$$

The last four terms in parentheses, which are small quantities of the second and third order, are disregarded; then, taking into account that $dx\,dy\,dz = d\tau$, we have

$$\delta(d\tau) = d\tau(e_{xx} + e_{yy} + e_{zz}).$$

Hence we obtain the dilatational strain (the cubical dilatation or simply the dilatation)

$$\frac{\delta(d\tau)}{d\tau} = e_{xx} + e_{yy} + e_{zz}.$$

Let this quantity be denoted by θ; in addition to equations (2.6) we write down the expression of the dilatational strain

$$\theta = e_{xx} + e_{yy} + e_{zz}, \tag{2.34}$$

or

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{2.35}$$

Thus, the dilatation coincides exactly with the first, linear, invariant of the strain tensor (2.32a). It is constructed, just as the other two invariants, similarly to the respective invariant of the stress tensor; therefore, all we have said in Sec. 5 concerning the invariants of the stress tensor may apply to the invariants of the strain tensor; this will reduce to the formal replacement of the notations

$$X_x, Y_y, Z_z, X_y, Y_z, Z_x$$

by the appropriate notations

$$e_{xx}, e_{yy}, e_{zz}, \bar{e}_{xy}, \bar{e}_{yz}, \bar{e}_{zx}.$$

*14. STRAIN DEVIATOR AND ITS INVARIANTS

The strain tensor

$$(\varepsilon) = \begin{Bmatrix} e_{xx} & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} & \bar{e}_{yz} \\ \bar{e}_{xz} & \bar{e}_{yz} & e_{zz} \end{Bmatrix} \tag{2.36}$$

is resolved, similarly to (2.28), into two tensors

$$(\varepsilon) = \begin{Bmatrix} e_{xx} - e & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} - e & \bar{e}_{yz} \\ \bar{e}_{xz} & \bar{e}_{yz} & e_{zz} - e \end{Bmatrix} + \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{2.37}$$

where

$$e = \frac{1}{3}(e_{xx} + e_{yy} + e_{zz}) \tag{2.38}$$

represents the mean elongation at a given point. The deformation corresponding to the second term in (2.37) is such that an element isolated at a given point in a body receives equal elongations in all directions (the strain surface is a sphere); hence, the element remains similar to itself, only its volume being changed,

and so the tensor

$$\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \tag{2.39}$$

is spherical. The first term

$$(D_e) = \begin{vmatrix} e_{xx} - e & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} - e & \bar{e}_{yz} \\ \bar{e}_{xz} & \bar{e}_{yz} & e_{zz} - e \end{vmatrix} \tag{2.40}$$

represents the *strain deviator* characterising the change of shape (shearing distortion or simply distortion) of the element isolated at a given point in a body without change in its volume, since in this case the volume expansion becomes zero

$$\theta = (e_{xx} - e) + (e_{yy} - e) + (e_{zz} - e) = e_{xx} + e_{yy} + e_{zz} - 3e = 0. \tag{2.41}$$

The resolution of (2.37) is not merely a formal operation; it reflects the physical properties of the phenomenon of deformation, since real materials behave differently, in the sense of resistance to the change of volume and to the change of shape.

Consider the invariants of the strain deviator; they are constructed in the same way as the invariants of the stress and strain tensors, with an appropriate replacement of notations.

The first, linear, invariant is zero, according to (2.41). The second, quadratic, invariant, by analogy with (1.49), has the form

$$\eta = -\frac{1}{6} [(e_{xx} - e_{yy})^2 + (e_{yy} - e_{yz})^2 + (e_{zz} - e_{xx})^2 + \\ + 6 (e_{xy} + e_{yz} + e_{zx})^2]; \tag{2.42}$$

it represents the first characteristic of distortion corresponding to the deformation defined by the strain tensor (2.36); if the principal axes of strain are taken for co-ordinate axes, then, according to (1.50), it will assume a simpler form

$$\eta = -\frac{1}{6} [(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2]. \tag{2.42'}$$

The third, cubic, invariant

$$\lambda' = \begin{vmatrix} e_{xx} - e & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} - e & \bar{e}_{yz} \\ e_{xz} & \bar{e}_{yz} & e_{zz} - e \end{vmatrix} \tag{2.43}$$

is the second characteristic of distortion; if it is referred to the principal axes of strain, it will assume, similarly to (1.51), a simpler form

$$\lambda' = (e_1 - e)(e_2 - e)(e_3 - e)$$

or, according to (1.52),

$$\lambda' = \frac{1}{3}[(e_1 - e)^3 + (e_2 - e)^3 + (e_3 - e)^3]. \tag{2.44}$$

Thus, it may be said (just as in the case of the stress deviator) that the second invariant of the strain deviator characterises the average quadratic deviation, and the third invariant—the average cubic deviation of the given deformation from the volumetric deformation defined by tensor (2.39), where

$$e = \frac{1}{3}(e_{xx} + e_{yy} + e_{zz}) = \frac{1}{3}(e_1 + e_2 + e_3)$$

and in which no distortion occurs. In this connection the quadratic invariants of the stress and strain deviators play an important role in the present theory of plasticity, since in the plastic state of a body the forces applied to it produce mainly a change in its shape and only an insignificant change in its volume.

In Chapter I we derived two basic groups of equations of the theory of elasticity: (1.5) and (1.8). These equations are of a static character. In the present chapter we have obtained new basic groups of equations of a geometrical character: (2.6), (2.15) and (2.35). It is to be noted that these equations are approximate since in deriving the basic relations

$$e_{xx} = \frac{\partial u}{\partial x}; \quad \alpha = \frac{\partial v}{\partial x}; \quad \beta = \frac{\partial u}{\partial y}$$

we have disregarded small but finite quantities. One must take it into consideration in evaluating all the subsequent conclusions; they will be sufficiently accurate in the case of small deformations (when all the strains $e_{xx}, e_{yy}, \ldots, e_{zx}, e_{xy}$ are very small as compared to unity); in the general case strains are expressed in terms of displacements in the form of far more complicated non-linear relations which will be dealt with in the next section.

*15. FINITE STRAIN

In the preceding sections, we dealt, strictly speaking, with infinitesimal deformations; indeed, basic relations (2.6) are absolutely exact in this case since only small quantities of higher order have been neglected in deriving these relations, but this

does not affect their accuracy. It will now be shown how to obtain more exact relations corresponding to finite deformations in place of relations (2.6).

Starting with formulas (2.23), we replace their left-hand members according to (2.21a, b) and rewrite them as

$$\begin{aligned}
\xi_1 &= \left(1+\frac{\partial u}{\partial x}\right)\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta,\\
\eta_1 &= \frac{\partial v}{\partial x}\xi + \left(1+\frac{\partial v}{\partial y}\right)\eta + \frac{\partial v}{\partial z}\zeta,\\
\zeta_1 &= \frac{\partial w}{\partial x}\xi + \frac{\partial w}{\partial y}\eta + \left(1+\frac{\partial w}{\partial z}\right)\zeta.
\end{aligned}$$

We divide both members of these equalities by the altered length of the segment ρ_1; furthermore, we multiply and divide the right-hand members by the original length of this segment ρ. Then, taking into account that

$$\frac{\xi}{\rho}=l;\quad \frac{\eta}{\rho}=m;\quad \frac{\zeta}{\rho}=n$$

represent the direction cosines of the segment ρ before deformation and

$$\frac{\xi_1}{\rho_1}=l_1;\quad \frac{\eta_1}{\rho_1}=m_1;\quad \frac{\zeta_1}{\rho_1}=n_1$$

are the direction cosines of this segment after deformation, we arrive at the following equatities:

$$\left.\begin{aligned}
l_1 &= \left[\left(1+\frac{\partial u}{\partial x}\right)l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n\right]\frac{\rho}{\rho_1},\\
m_1 &= \left[\frac{\partial v}{\partial x}l + \left(1+\frac{\partial v}{\partial y}\right)m + \frac{\partial v}{\partial z}n\right]\frac{\rho}{\rho_1},\\
n_1 &= \left[\frac{\partial w}{\partial x}l + \frac{\partial w}{\partial y}m + \left(1+\frac{\partial w}{\partial z}\right)n\right]\frac{\rho}{\rho_1}.
\end{aligned}\right\} \tag{2.45}$$

We eliminate from these equalities the direction cosines l_1, m_1, n_1 of the segment ρ_1, which are obtained in the process of deformation and are not known in advance; for this purpose we square all three equalities and add them up by members; then, taking into account that

$$l_1^2+m_1^2+n_1^2=1,$$

we obtain the relation

$$\left(\frac{\rho_1}{\rho}\right)^2 = A^2+B^2+C^2, \tag{2.46}$$

where A, B, C represent the bracketed expressions in equalities (2.45). Writing out the values of A^2, B^2, C^2, we get

$$A^2 = l^2 + 2\frac{\partial u}{\partial x} l^2 + \left(\frac{\partial u}{\partial x}\right)^2 l^2 + \left(\frac{\partial u}{\partial y}\right)^2 m^2 + \left(\frac{\partial u}{\partial z}\right)^2 n^2 +$$
$$+ 2\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) lm + 2\left(\frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) ln + 2\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} mn,$$

$$B^2 = m^2 + 2\frac{\partial v}{\partial y} m^2 + \left(\frac{\partial v}{\partial y}\right)^2 m^2 + \left(\frac{\partial v}{\partial x}\right)^2 l^2 + \left(\frac{\partial v}{\partial z}\right)^2 n^2 +$$
$$+ 2\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) lm + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial z} ln + 2\left(\frac{\partial v}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) mn,$$

$$C^2 = n^2 + 2\frac{\partial w}{\partial z} n^2 + \left(\frac{\partial w}{\partial z}\right)^2 n^2 + \left(\frac{\partial w}{\partial x}\right)^2 l^2 + \left(\frac{\partial w}{\partial y}\right)^2 m^2 + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} lm +$$
$$+ 2\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) ln + 2\left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) mn.$$

We add up these three expressions together and substitute them in (2.46); we begin with the first terms of the expressions and note that

$$l^2 + m^2 + n^2 = 1;$$

we transpose this result to the left-hand side; adding up the remaining terms of the expressions, we collect together the coefficients of the squares and products of the cosines l, m, n. We transform the left-hand member of the equality as

$$\left(\frac{\rho_1}{\rho}\right)^2 - 1 = \left(\frac{\rho_1 - \rho}{\rho} + 1\right)^2 - 1 = (e+1)^2 - 1 = e^2 + 2e;$$

here $e = \frac{\rho_1 - \rho}{\rho}$ represents the unit elongation of the segment ρ. Hence we obtain the following equality:

$$e^2 + 2e = 2(\varepsilon_{xx} l^2 + \varepsilon_{yy} m^2 + \varepsilon_{zz} n^2 + 2\bar{\varepsilon}_{xy} lm + 2\bar{\varepsilon}_{yz} mn + 2\bar{\varepsilon}_{zx} nl), \quad (2.47)$$

where for brevity we have introduced the notation

$$\left.\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right], \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right], \\
2\bar{\varepsilon}_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \\
2\bar{\varepsilon}_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}, \\
2\bar{\varepsilon}_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}.
\end{aligned}\right\} \quad (2.48)$$

Comparing relation (2.47) with (1.19) and (2.27), we see that quantities (2.48) are the components of the tensor

$$\begin{Vmatrix} \varepsilon_{xx} & \bar{\varepsilon}_{xy} & \bar{\varepsilon}_{xz} \\ \bar{\varepsilon}_{xy} & \varepsilon_{yy} & \bar{\varepsilon}_{yz} \\ \bar{\varepsilon}_{xz} & \bar{\varepsilon}_{yz} & \varepsilon_{zz} \end{Vmatrix}. \tag{2.49}$$

To distinguish this tensor from (2.36), it is called the finite strain tensor. If its components are known, we can calculate the unit elongations at a given point in a body in any direction defined by the cosines l, m, n; indeed, denoting for brevity the right-hand member of (2.47) by $2f(l, m, n)$, we obtain the quadratic equation for the determination of e

$$e^2+2e-2f=0; \quad e=\sqrt{1+2f}-1; \tag{2.50}$$

the second root is dropped, since it gives only a negative value of e. The first terms in formulas (2.48) coincide with the right-hand members of Cauchy's formulas (2.6) which are obtainable from those formulas if we disregard the squares and products of the components of the tensor of relative displacements (2.17) in the case of their smallness. It should be noted, however, that components (2.48) do not by themselves express strains as contrasted to the case of small deformations. Indeed, let us take the segment ρ in the direction of the x axis, setting $l=1$, $m=n=0$; then from (2.47) we find the elongation along the x axis

$$e_{xx}=\sqrt{1+2\varepsilon_{xx}}-1.$$

If components (2.7) represent proper fractions considerably less than unity, then assuming approximately

$$\sqrt{1+2\varepsilon_{xx}} \approx 1+\varepsilon_{xx},$$

we find that they approximate to strains, for instance,

$$e_{xx} \approx \varepsilon_{xx}.$$

Finally, if all the derivatives entering in (2.48) are very small, then, disregarding small quantities of the second order in (2.48), we return to Cauchy's formulas.

Exercises to Sections 10 and 11.

1. Given a rod subjected to bending in the plane Oxz (Fig. 8), it is required:

(a) to write the equation of its axis (i.e., axis Ox) before bending and to express the deflection of any point by the use of equations (2.2) of Sec. 10;

(b) assuming that the rod is built-in at the origin and using equations (2.3) and (2.4), to write down the conditions that the element dx of the axis

of the rod at the origin is fixed (does not rotate in the plane Oxz); the same for the element dz in the plane of the cross section.

2. In the case of the pure bending of the rod considered in the preceding exercise, displacements (2.2) of Sec. 10 are expressed as

$$\left.\begin{aligned} u &= -\frac{xz}{\rho}, \\ v &= \frac{\sigma yz}{\rho}, \\ w &= \frac{x^2+\sigma(z^2-y^2)}{2\rho}. \end{aligned}\right\} \tag{2.51}$$

It is required:

(a) to find the general expressions of strains according to equations (2.6) and (2.35) and to check whether they satisfy the compatibility equations (2.15);

(b) to write the equation of the deflected axis of the rod; for that substitute the co-ordinates of a point of the axis before bending in expressions (2.51) (point "a" of the preceding exercise);

(c) to check whether the fastening conditions at the left-hand end are satisfied (point "b" of the preceding exercise).

3. In an investigation of the torsion of a circular bar (Fig. 7) one obtains the displacements

$$u = -\tau yz + ay + bz + c; \quad v = \tau xz - ax + ez + f; \quad w = -bx - ey + k.$$

It is required:

(a) to select the coefficients a, b, c, e, f, k so that the end section $z=0$ will be fixed in the following manner: the point O must have no displacements; the element of the axis dz rotates neither in the plane xOz nor in the plane yOz; the element of the axis dy does not rotate in the plane xOy;

(b) to find the magnitudes of the strains (2.6) and (2.35);

(c) to check whether equations (2.15) are satisfied.

III

Generalised Hooke's Law

16. GENERAL

In the preceding chapters we presented the theory of stresses elucidating the static aspect of the problem, and also the theory of displacements and strains elucidating the problem from the geometrical point of view. These two theories alone cannot be instrumental in solving the physical problems of the theory of elasticity concerning deformations that are produced in an elastic body under the action of external forces applied to it until stresses and strains have been connected by a *physical* law. The physical nature of this law makes it connect *dissimilar* features of the phenomenon under investigation—stresses and strains.

The theory of elasticity treats this law in the most general form. Its analytic form, i.e., the general form of functions relating stresses to strains, is ascertained in the first place:

$$\left.\begin{aligned} X_x &= f_1(e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy}); \\ Y_y &= f_2(e_{xx}, e_{yy}, \ldots\ldots, e_{xy}); \\ Z_z &= f_3(e_{xx}, e_{yy}, \ldots\ldots, e_{xy}); \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ X_y &= f_6(e_{xx}, e_{yy}, \ldots\ldots, e_{xy}): \end{aligned}\right\} \tag{3.1}$$

it is readily established that in the case of small deformations the simplest and most rational form of relations (3.1) is a linear form. After that the values of the coefficients of functions (3.1) are determined for various cases of structure of an elastic body. Denoting the coefficients by a_{mn}, we represent the first function of (3.1) as follows:

$$X_x = a_{11}e_{xx} + a_{12}e_{yy} + a_{13}e_{zz} + a_{14}e_{yz} + a_{15}e_{zx} + a_{16}e_{xy}. \tag{3.2}$$

The linear form of relations (3.1) formulates in its very essence the *principle of superposition* (the law of independence of effects

of forces) implying, for example, that the stress X_x developed in the presence of several strain components e_{xx}, e_{yy}, ..., e_{xy} is equal to the sum of stresses caused by each of these components separately.

According to their structure, physical bodies are divided, first of all, into *homogeneous* and *nonhomogeneous*. A homogeneous body is one whose structure and composition are the same at all its points. The theory of elasticity deals almost exclusively with homogeneous bodies; however, even among homogeneous bodies one has to distinguish between *isotropic* bodies whose properties are the same in all directions and *nonisotropic* (anisotropic) bodies. Many crystals appear to be nonisotropic; while

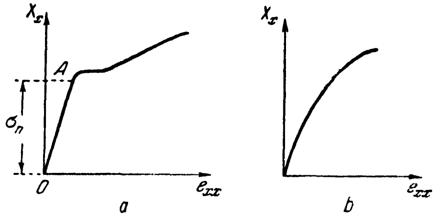

Fig. 27

the structure of such crystals is homogeneous, their elastic and optical properties are different in different directions. A detailed analysis of relations such as (3.2) for nonisotropic bodies shows that the numerical values of the coefficients a_{mn} are closely related to the elastic properties of a given body in different directions.

It follows from this analysis that relations (3.2) assume the simplest form for an isotropic elastic body, i.e., for a body whose physical properties are identical in all directions; they can be derived on the basis of Hooke's law for elastic bars in tension and compression, which is known from physics, and also on the basis of the above-formulated principle of superposition.

Fig. 27*a*, *b* shows the so-called tensile test diagram expressing the relation between the tensile (normal) stress X_x developed in the bar and its unit elongation e_{xx}. The shape of this diagram is different for different materials and is substantially affected by their chemical composition and structure. A diagram of the *a* type is characteristic of metals possessing ductile properties, for instance, of mild steels with low content of carbon. The initial portion *OA* of this diagram is straight, provided the test is conducted thoroughly, an indication of the proportionality between stress and

unit elongation

$$X_x = Ee_{xx}; \tag{3.3a}$$

this relation formulates Hooke's law of elasticity. The terminal point A of the portion corresponds to the stress σ_p which is called the *proportional limit*. At the same time, experiments show that for nearly all materials the longitudinal elongation of the bar e_{xx} in simple tension is accompanied by equal lateral deformations $e_{yy}=e_{zz}$ of opposite sign (i.e., by contractions), these deformations being proportional to the primary elongation e_{xx}:

$$e_{yy} = e_{zz} = -\sigma e_{xx}; \tag{3.3b}$$

the factor of proportionality σ represents a constant for each material but it is different for different materials. Thus, Hooke's law is formulated by two relations: (3.3a) and (3.3b); they contain two numbers characterising the elastic properties of a material: the longitudinal modulus of elasticity or Young's modulus E (also called the tensile or compressive modulus of elasticity or simply the modulus of elasticity) having the dimension of stress |force/length²| and the dimensionless number σ called *Poisson's ratio*; sometimes the reciprocal of Poisson's ratio, $m=\frac{1}{\sigma}$, is employed. It is called *Poisson's number*.

The numerical value of the modulus of elasticity E varies over a very wide range for different materials; for example, for steels we have approximately $E=2.1\times10^6$ kg/cm², for wood $E=1\times\times10^5$ kg/cm². Poisson's ratio σ is always expressed by a proper fraction smaller than 0.5; this latter circumstance can be established a priori from physical considerations, as will be shown in Sec. 18. In the case of materials which possess ductile properties to a small extent only or not at all, i.e., in the case of brittle materials, such, for instance, as hard alloy steels, cast iron, stone, the tensile test diagram has no initial straight-line portion (Fig. 27*b*); but in most cases its initial portion deviates but slightly from a straight line; to simplify the theory this portion is replaced approximately by a straight line and thus Hooke's law is sometimes applied conditionally to materials which behave as brittle. Experiments show that as long as the material acts under conditions of elastic properties (the straight-line portion of the diagram in Fig. 27*a*), there is proportionality between shearing stresses on the faces of an elementary parallelepiped and shearing strains:

$$X_y = Ge_{xy}; \quad Y_z = Ge_{yz}; \quad Z_x = Ge_{zx}.$$

This relation formulates *Hooke's law in shear*; the factor of proportionality G is called the *modulus of elasticity in shear* or the

transverse modulus of elasticity (also called the shearing modulus of elasticity, the shear modulus or modulus of rigidity); its dimension, just as the dimension of the tensile (compressive) modulus of elasticity, coincides with the dimension of stress. The modulus of elasticity in shear is also a characteristic of the elastic properties of a material but, as we shall see at the end of Sec. 23, there exists a relation between the three elastic constants E, G and σ

$$G=\frac{E}{2(1+\sigma)} \tag{3.4}$$

resulting from Hooke's law; therefore, only two of these constants are independent and they must be found by experiment; the third constant is determined from formula (3.4). For instance, it is most convenient and reliable to determine the moduli E and G by experiments; then Poisson's ratio is calculated according to formula (3.4):

$$\sigma=\frac{E}{2G}-1.$$

In addition to considerations presented here, let us assume, as yet without proof, that in an elastic homogeneous isotropic material normal stresses do not cause shear deformations and, vice versa, shearing stresses do not produce elongations in the direction of their action. On the basis of this assumption we may consider separately the case of normal stresses and the case of shearing stresses to obtain the most general relations between stresses and strains of an elastic homogeneous isotropic material. A better substantiated derivation of these relations and their extension to homogeneous but anisotropic materials will be given in Sections 19-23.

17. STRAINS EXPRESSED IN TERMS OF STRESSES

Let us begin with normal stresses and consider (Fig. 28) a parallelepiped with edges equal to unity, subjected to the action of normal forces. If the stress X_x were acting alone, the unit elongation e_{xx}, according to Hooke's law, would be $e'_{xx}=\frac{X_x}{E}$. Under the action of the stress Y_y alone, the elongation along the axis Ox would be $e''_{xx}=-\sigma\frac{Y_y}{E}$, where σ is Poisson's ratio. Likewise, under the action of the stress Z_z we would have $e'''_{xx}=-\sigma\frac{Z_z}{E}$.

Assuming without proof the principle of superposition, we obtain the total elongation along the axis Ox:

$$e_{xx} = e'_{xx} + e''_{xx} + e'''_{xx} = \frac{X_x}{E} - \sigma\frac{Y_y}{E} - \sigma\frac{Z_z}{E},$$

or

$$e_{xx} = \frac{1}{E}[X_x - \sigma(Y_y + Z_z)].$$

Similarly, for elongations along the other two axes we have

$$e_{yy} = \frac{1}{E}[Y_y - \sigma(Z_z + X_x)]; \quad e_{zz} = \frac{1}{E}[Z_z - \sigma(X_x + Y_y)]. \quad (3.5)$$

Formulas (3.5) express the generalised Hooke's law for normal

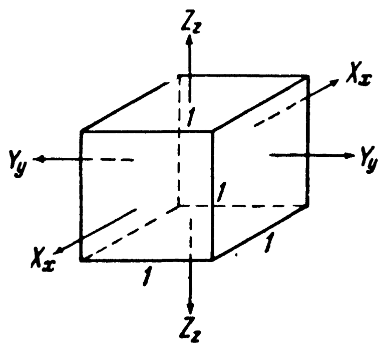

Fig. 28

stresses; they can be represented in the alternate form adding to the right-hand members respectively

$$\sigma X_x - \sigma X_x; \quad \sigma Y_y - \sigma Y_y; \quad \sigma Z_z - \sigma Z_z;$$

we then obtain

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E}[(1+\sigma)X_x - \sigma\Theta], \\ e_{yy} &= \frac{1}{E}[(1+\sigma)Y_y - \sigma\Theta], \\ e_{zz} &= \frac{1}{E}[(1+\sigma)Z_z - \sigma\Theta], \end{aligned}\right\} \quad (3.6)$$

where

$$\Theta = X_x + Y_y + Z_z. \quad (3.7)$$

Hooke's law is more convenient in this form for some computations.

Relations between shearing strains and stresses are taken in the above-indicated form:

$$\left.\begin{aligned} e_{yz} &= \frac{1}{G} Y_z, \\ e_{zx} &= \frac{1}{G} Z_x, \\ e_{xy} &= \frac{1}{G} X_y. \end{aligned}\right\} \tag{3.8}$$

Formulas (3.5) or (3.6) in conjunction with formulas (3.8) give the *generalised Hooke's law* for a homogeneous and isotropic elastic body, i.e., a body whose elastic properties are identical in all directions.

This leads us to a new basic group of equations of the theory of elasticity

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E} [X_x - \sigma (Y_y + Z_z)]; \\ e_{yy} &= \frac{1}{E} [Y_y - \sigma (Z_z + X_x)]; \\ e_{zz} &= \frac{1}{E} [Z_z - \sigma (X_x + Y_y)]; \\ e_{yz} &= \frac{1}{G} Y_z = \frac{2(1+\sigma)}{E} Y_z; \\ e_{zx} &= \frac{1}{G} Z_x = \frac{2(1+\sigma)}{E} Z_x; \\ e_{xy} &= \frac{1}{G} X_y = \frac{2(1+\sigma)}{E} X_y. \end{aligned}\right\} \tag{3.9}$$

Let us revert to equalities (3.6) and add them up by members:

$$e_{xx} + e_{yy} + e_{zz} = \frac{1}{E} [(1+\sigma)(X_x + Y_y + Z_z) - 3\sigma\Theta].$$

Since on the basis of equations (3.7) and (2.34)

$$X_x + Y_y + Z_z = \Theta; \quad e_{xx} + e_{yy} + e_{zz} = \theta,$$

the last equation can finally be written down as

$$\theta = \frac{1-2\sigma}{E} \Theta, \tag{3.10}$$

i.e., the unit volume expansion θ is proportional to the sum of three normal stresses Θ. This is Hooke's law in the volumetric form.

From equality (3.10) we can draw the following conclusion: the dilatation θ as such must not depend on the directions of the co-ordinate axes. Consequently, it does not vary with the rotation of the co-ordinate system and, therefore, according to (3.10), *the*

sum of normal stresses on three mutually perpendicular areas at a given point is a constant quantity and does not depend on the inclination of these areas. In other words, the quantity

$$\Theta = X_x + Y_y + Z_z$$

is an invariant under a transformation of co-ordinates, as already stated in Chapter I.

18. STRESSES EXPRESSED IN TERMS OF STRAINS

Equalities (3.9) and (3.10) give Hooke's law in the form solved with respect to strains. It is often necessary to have the same relations in the inverse form, i.e., in the form solved with respect to stresses. For this purpose we take the first of equations (3.6) and introduce in it the value $\Theta = \frac{E}{1-2\sigma}\theta$ from equation (3.10); we get

$$e_{xx} = \frac{1}{E}\left[(1+\sigma)X_x - \frac{E\sigma}{1-2\sigma}\theta\right];$$

hence we find

$$X_x = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\theta + \frac{E}{1+\sigma}e_{xx},$$

or

$$X_x = \lambda\theta + 2\mu e_{xx}, \tag{3.11}$$

where

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}; \qquad \mu = \frac{E}{2(1+\sigma)}. \tag{3.12}$$

λ and μ are called Lamé's coefficients. They characterise, as do the moduli of elasticity E and G, the elastic properties of a solid, being connected with the moduli of elasticity by formulas (3.12).

It can easily be observed that the coefficient μ coincides with the modulus of shear $\mu = \frac{E}{2(1+\sigma)} = G$.

From equation (3.11) we obtain two other equations by a cyclic change; adding to them the last three equations of group (3.9), we have Hooke's law in the desired form solved with respect to stresses

$$\left.\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx}, \\ Y_y &= \lambda\theta + 2\mu e_{yy}, \\ Z_z &= \lambda\theta + 2\mu e_{zz}, \\ Y_z &= \mu e_{yz}, \\ Z_x &= \mu e_{zx}, \\ X_y &= \mu e_{xy}. \end{aligned}\right\} \tag{3.13}$$

Summing up the first three equations of (3.13) by members, we again get the volumetric Hooke's law but expressed now in terms of Lame's coefficients:

$$X_x + Y_y + Z_z = 3\lambda\theta + 2\mu(e_{xx} + e_{yy} + e_{zz}),$$

or

$$\Theta = (3\lambda + 2\mu)\,\theta. \tag{3.14}$$

Dividing both members of this equality by 3 and taking into account that (Sections 7 and 13)

$$\frac{1}{3}\Theta = N_0$$

represents the mean or octahedral [1] stress and

$$\frac{1}{3}\theta = e$$

is the mean elongation at a given point, we represent relation (3.14) as

$$N_0 = (3\lambda + 2\mu)\,e. \tag{3.15}$$

Formulas (3.12) express Lamé's coefficients in terms of the modulus E and Poisson's ratio σ; we can easily obtain inverse relationships; for that we eliminate the modulus E from formulas (3.12) by dividing them by members; we get

$$\frac{\lambda}{\mu} = \frac{2\sigma}{1 - 2\sigma};$$

hence

$$\sigma = \frac{\lambda}{2(\lambda + \mu)}$$

and further

$$1 + \sigma = \frac{3\lambda + 2\mu}{2(\lambda + \mu)}.$$

Substituting this in the second of formulas (3.12), we find

$$E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}.$$

Thus, we have the formulas expressing E and σ in terms of Lamé's coefficients:

$$E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}; \quad \sigma = \frac{\lambda}{2(\lambda + \mu)}. \tag{3.16}$$

We have considered two possible ways of expressing the generalised Hooke's law: in terms of the moduli E and σ and in terms of Lamé's coefficients λ and μ.

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[1] See Sec. 7.

It is possible to introduce two other constants for an isotropic body in different ways; it is sometimes convenient, for example, to introduce the following two constants:

$$\left.\begin{aligned} \mu &= G, \\ k &= \frac{\lambda}{\mu} = \frac{2\sigma}{1-2\sigma}; \end{aligned}\right\} \tag{3.17}$$

then equations (3.13) are written down as

$$\left.\begin{aligned} X_x &= \mu(2e_{xx} + k\theta), \\ Y_y &= \mu(2e_{yy} + k\theta), \\ Z_z &= \mu(2e_{zz} + k\theta), \\ Y_z &= \mu e_{yz}, \\ Z_x &= \mu e_{zx}, \\ X_y &= \mu e_{xy}. \end{aligned}\right\} \tag{3.18}$$

If the element under consideration undergoes extension in the directions of the three axes x, y, z, i.e., if

$$X_x > 0; \quad Y_y > 0; \quad Z_z > 0; \quad \Theta > 0,$$

its volume must not decrease and we must have

$$\theta \geqslant 0.$$

On this basis we obtain from (3.10)

$$\sigma \leqslant \frac{1}{2},$$

but then from (3.12) and (3.17) we have

$$\lambda > 0; \quad \mu > 0; \quad k > 0.$$

Relations (3.9) and (3.13) developed in this chapter constitute the last group of basic equations of the theory of elasticity, which completes all the necessary premises for the general solution of the elasticity problem.

We have assumed without proof the equations of the generalised Hooke's law in the form they are given in strength of materials. Sections 18-23 below present considerations to prove that these equations give the most general relation between stresses and strains in an isotropic elastic body.

In conclusion it may be noted that the concepts of the stress and strain tensors and deviators introduced in Sections 4, 8, 12 and 14 permit us to express the generalised Hooke's law in more compact tensor form. Indeed, let us develop the expressions of the

components of the stress deviator (1.47) in terms of strains by using relations (3.13). Taking into account relationship (3.15), we get

$$X_x - N_0 = 3\lambda e + 2\mu e_{xx} - (3\lambda + 2\mu) e = 2\mu (e_{xx} - e);$$

in the same way we find the other two normal stresses:

$$Y_y - N_0 = 2\mu (e_{yy} - e),$$
$$Z_z - N_0 = 2\mu (e_{zz} - e).$$

The shearing stresses (expressed through the halves of the shear angles) are

$$Y_z = 2\mu \bar{e}_{yz},$$
$$Z_x = 2\mu \bar{e}_{zx},$$
$$X_y = 2\mu \bar{e}_{xy}.$$

Examining these relations, we see that all the components of the stress deviator are proportional to the corresponding components of the strain deviator with one and the same factor of proportionality 2μ. This fact can be expressed in tensor form: *the stress deviator is proportional to the strain deviator*

$$\begin{pmatrix} X_x - N_0 & X_y & X_z \\ Y_x & Y_y - N_0 & Y_z \\ Z_x & Z_y & Z_z - N_0 \end{pmatrix} = 2\mu \begin{pmatrix} e_{xx} - e & \bar{e}_{xy} & \bar{e}_{xz} \\ \bar{e}_{xy} & e_{yy} - e & \bar{e}_{yz} \\ \bar{e}_{xz} & \bar{e}_{yz} & e_{zz} - e \end{pmatrix}$$

or in the compact notation of the deviators accepted by us

$$(D_s) = 2\mu (D_e). \tag{3.19}$$

In the left-hand member of equality (3.14) expressing the volumetric Hooke's law, we introduce the mean (hydrostatic) stress in place of Θ:

$$\Theta = 3N_0;$$

we then obtain

$$N_0 = K\theta, \tag{3.20}$$

where

$$K = \lambda + \frac{2}{3}\mu$$

is termed the *volumetric modulus of elasticity* or the *bulk modulus* (also called the modulus of volume expansion or the modulus of compressibility of the material). We see, thus, that the generalised Hooke's law is expressible by two equalities—scalar (3.20) and tensor (3.19)—and contains, as above, two elastic constants 2μ and K.

Exercises.

The reader is recommended to carry out the following derivations, as they are useful for what follows.

1. A thin rectangular rubber plate rests closely, but without pressure, between two steel plates which can be assumed to be absolutely rigid as compared to the rubber plate. Friction between the plates is eliminated. If the co-ordinate system is chosen so that the axis Oz is normal to the planes of contact of the plates, then, according to the condition of the problem, one may assume that there are no elongations along the axis Oz, i.e., $e_{zz}=0$. The rubber plate is compressed by the forces applied to the faces normal to the axes Ox and Oy.

It is required to find the relation between normal stresses and elongations, and also the relation among stresses X_x, Y_y, Z_z. Solve the problem in two variants, starting with equations (3.9) and (3.13).

2. A rubber cube is inserted tightly in a steel box of the same form; the cube is closely covered at the top with a steel plate, to which pressure p kg/cm² is applied. Considering the steel to be absolutely hard and assuming that there is no friction between steel and rubber, find:

(a) the pressure of rubber against the box walls;

(b) the maximum shearing stresses in rubber [starting with equations (1.42) of Sec. 6].

Solve the problem in two variants starting with equations (3.9) and (3.13), carry out calculations for cases $\sigma=0$ and $\sigma=\frac{1}{2}$, find θ and analyse both of these cases.

*19. WORK DONE BY ELASTIC FORCES IN A SOLID

Let us isolate an infinitesimal parallelepiped of dimensions dx, dy, dz in a body and calculate the work done by the elastic forces applied to it, if we give some virtual displacements to points of the given body.

First consider the case of tension-compression of the element in the direction of the axis Ox. Let the stresses on the faces normal to Ox be

$$X_x \text{ and } X_x+\frac{\partial X_x}{\partial x}dx \tag{3.21}$$

with the resulting unit elongation e_{xx} and, consequently, absolute elongation $e_{xx}dx$. By giving some virtual displacements to the body this elongation will be changed by a certain amount $\delta e_{xx}dx$.

Disregarding in (3.21) the infinitesimal $\frac{\partial X_x}{\partial x}dx$ as compared with the finite quantity X_x, we find that two equal and opposite normal forces $X_x\,dy\,dz$ applied to the faces of the parallelepiped do the work $X_x\delta e_{xx}dx\,dy\,dz$. In the same way we obtain the work done by the remaining normal forces acting on the faces of the parallelepiped:

$$Y_y\delta e_{yy}\,dy\,dx\,dz;\quad Z_z\delta e_{zz}\,dz\,dx\,dy.$$

We proceed to the tangential components. The upper and lower faces of the parallelepiped (Fig. 10) are acted upon by the tangential tractions $X_z dx\,dy$ (the infinitesimal difference between them is again neglected) forming a couple with the moment

$$X_z\,dx\,dy\,dz;$$

to obtain the work done by this couple we must multiply its moment by the angle of rotation; this angle of rotation will obviously be given by the increment

$$\delta e_{xz}$$

which the shear angle e_{xz} receives on the virtual displacement undergone by the body. On this basis we obtain the work done by three pairs of tangential tractions applied to the parallelepiped:

$$X_z\,\delta e_{xz}\,dx\,dy\,dz; \quad Y_z\,\delta e_{yz}\,dz\,dx\,dy; \quad X_y\,\delta e_{xy}\,dy\,dx\,dz.$$

Collecting together the results obtained, we write down the work done by all the forces applied to the parallelepiped on the virtual displacement:

$$\delta(dT) = (X_x\,\delta e_{xx} + Y_y\,\delta e_{yy} + Z_z\,\delta e_{zz} + Y_z\,\delta e_{yz} + \\ + Z_x\,\delta e_{zx} + X_y\,\delta e_{xy})\,dx\,dy\,dz.$$

The work per unit volume of the body at the point where the parallelepiped has been isolated is

$$\delta A = \frac{\delta(dT)}{d\tau} = X_x\,\delta e_{xx} + Y_y\,\delta e_{yy} + Z_z\,\delta e_{zz} + Y_z\,\delta e_{yz} + \\ + Z_x\,\delta e_{zx} + X_y\,\delta e_{xy}. \quad (3.22)$$

*20. POTENTIAL OF ELASTIC FORCES

We now introduce a hypothesis which is exceedingly important in the theory of elasticity, stating that internal elastic forces have a potential, i.e., the work done by elastic forces is transformed completely into the potential energy accumulated in a body when it undergoes elastic deformations and given up by the body in the form of work done by the forces when the deformations disappear [1]. The potential energy is due to deformations and is caused by them

[1] This hypothesis is only approximately true, for a part of the work done by elastic forces is transformed into other forms of energy (thermal and electromagnetic), which are lost by the body and are not recovered in the form of work done by elastic forces; these additional amounts of energy, however, are not great and may be ignored in most practical cases.

alone; therefore, if we define by W the energy per unit volume of a body at a given point, it must be a function of the components of strain:

$$W = F(e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy}). \tag{3.23}$$

If we give elastic virtual displacements to a body, then, according to the hypothesis introduced, the work (3.22) done by internal forces on these displacements must be transformed completely into elastic energy and give an increment δW:

$$\delta A = \delta W. \tag{3.24}$$

The increment δW of function (3.23) is replaced by its first total differential, to small quantities of the second order,

$$\delta W = \frac{\partial W}{\partial e_{xx}} \delta e_{xx} + \frac{\partial W}{\partial e_{yy}} \delta e_{yy} + \ldots + \frac{\partial W}{\partial e_{xy}} \delta e_{xy}. \tag{3.25}$$

Expressions (3.22) and (3.25), according to equation (3.24), must be coincident for any values of virtual deformations

$$\delta e_{xx}, \delta e_{yy}, \delta e_{zz}, \ldots, \delta e_{xy}. \tag{3.26}$$

By virtue of this, the coefficients of quantities (3.22) in equations (3.22) and (3.25) must be identically equal:

$$\left.\begin{aligned} X_x &= \frac{\partial W}{\partial e_{xx}}; \quad Y_y = \frac{\partial W}{\partial e_{yy}}; \quad Z_z = \frac{\partial W}{\partial e_{zz}}; \\ Y_z &= \frac{\partial W}{\partial e_{yz}}; \quad \ldots; \quad X_y = \frac{\partial W}{\partial e_{xy}}. \end{aligned}\right\} \tag{3.27}$$

Thus, if internal elastic forces have a potential, the *stress components*

$$X_x, Y_y, Z_z, Y_z, Z_x, X_y$$

are expressed as partial derivatives of the potential energy (3.23) *with respect to corresponding deformations.* Equalities (3.27) in conjunction with (3.23) are an analytic expression of the assumption concerning the existence of the potential of elastic forces.

*21. STRESS-STRAIN RELATIONS; HYPOTHESIS OF THE NATURAL STATE OF A BODY

Since we are seeking a relation of a physical nature, the external mathematical form of functions $f_1, f_2, \ldots, f_6$ in equations (3.1) (Sec. 16), expressing this relation, is in no way restricted in advance and we have the right to choose it as we please, provided it corresponds to the physical conditions of the problem, i.e., provided it reflects correctly the physical phenomenon under

investigation. Taking advantage of this, we select the simplest form of relations (3.1), viz., a linear form:

$$\left.\begin{aligned} X_x &= a_{11}e_{xx} + a_{12}e_{yy} + a_{13}e_{zz} + a_{14}e_{yz} + a_{15}e_{zx} + a_{16}e_{xy}, \\ Y_y &= a_{21}e_{xx} + a_{22}e_{yy} + a_{23}e_{zz} + a_{24}e_{yz} + a_{25}e_{zx} + a_{26}e_{xy}, \\ Z_z &= a_{31}e_{xx} + a_{32}e_{yy} + a_{33}e_{zz} + a_{34}e_{yz} + a_{35}e_{zx} + a_{36}e_{xy}, \\ Y_z &= a_{41}e_{xx} + a_{42}e_{yy} + a_{43}e_{zz} + a_{44}e_{yz} + a_{45}e_{zx} + a_{46}e_{xy}, \\ Z_x &= a_{51}e_{xx} + a_{52}e_{yy} + a_{53}e_{zz} + a_{54}e_{yz} + a_{55}e_{zx} + a_{56}e_{xy}, \\ X_y &= a_{61}e_{xx} + a_{62}e_{yy} + a_{63}e_{zz} + a_{64}e_{yz} + a_{65}e_{zx} + a_{66}e_{xy}. \end{aligned}\right\} \quad (3.28)$$

The considerations supporting the reasonableness of our choice are as follows.

1. The linear form of the relations is in good agreement with experiments for many materials under simple tension or compression, where it is observed as direct proportionality between stress and unit elongation (Hooke's law).

2. To given strains

$$e_{xx}, \; e_{yy}, \; e_{zz}, \; e_{yz}, \; e_{zx}, \; e_{xy} \quad (3.29)$$

at a given point there should correspond just one system of stresses

$$X_x, \; Y_y, \; Z_z, \; Y_z, \; Z_x, \; X_y \quad (3.30)$$

and, conversely, given stresses (3.30) entail exactly one system of strains (3.29).

These conditions are satisfied by a linear form of the relations [1].

3. If we limit ourselves, as before, to a case of very small deformations, we shall show that whatever the analytic form of relations (3.1) may be, they can be replaced approximately by linear functions; in fact, expanding the right-hand members of equations (3.1) in a Maclaurin series, we obtain, to small quantities of the second order,

$$X_x = (f_1)_0 + \left(\frac{\partial f_1}{\partial e_{xx}}\right)_0 e_{xx} + \left(\frac{\partial f_1}{\partial e_{yy}}\right)_0 e_{yy} + \ldots + \left(\frac{\partial f_1}{\partial e_{xy}}\right)_0 e_{xy}, \quad (3.31)$$

where the zero index in the function f_1 and its derivatives shows that one must put

$$e_{xx} = e_{yy} = \ldots = e_{xy} = 0.$$

[1] If we solve equations (3.28) with respect to strains, we shall also obtain linear expressions for them in terms of stresses.

The coefficients of the strains in (3.31) are constant numbers and, therefore, function (3.31) is linear with respect to strains (3.29) [1].

It should also be noted that in the right-hand members of equations (3.28) we have omitted constant terms that are not dependent on strains. If we retain them and set all the strains (3.29) equal to zero, we shall find that stresses (3.28) are different from zero (stresses produce no deformations); conversely, setting all the stresses (3.30) equal to zero, we convert equations (3.28) into a system of six linear equations with six unknowns (3.29) for which we find nonzero values (deformations are not accompanied by stresses).

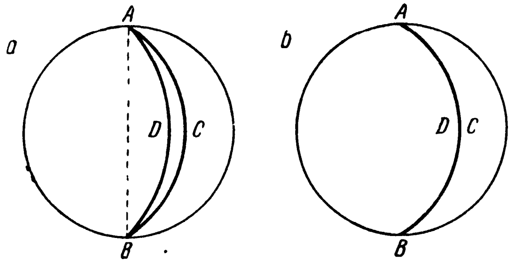

Fig. 29

Both circumstances may occur, since in studying deformations caused in a body by external forces, it may sometimes be expected that the body has undergone deformation prior to the application of forces. For instance, take a piece of iron in the form of a sphere (Fig. 29*a*) and, making a narrow sectorial cut-out *ACBD* in it, join again the cut planes *ACB* and *ADB* and weld them. We obtain a solid body (Fig. 29*b*): after this operation there will undoubtedly be induced stresses in it, although no external forces act upon it; such stresses are called *initial*; they exist, for instance, in cast iron and steel castings on account of nonuniform shrinkage in the process of cooling. Let us take state *b* of the body as initial when measuring deformations under the action of some forces; we say, then, that *there are no deformations in this state but the stresses are not equal to zero.* If we cut the body again through the plane *ACB*, it will return to its former state *a*; *deformations will occur then, but the stresses will become zero.*

[1] This reasoning holds true unless all the coefficients of the strains in equation (3.31) vanish, in which case one must take into consideration small terms of the second order in the Maclaurin series; no such case is possible, however, since in the case of simple tension the existence of first order terms is readily ascertained (Hooke's law).

The nature and magnitude of initial stresses are usually unknown because they depend upon the prior history of a body [1]; henceforth we shall exclude them from consideration, *introducing the hypothesis of the natural state of a body*, i.e., assuming that *in the absence of deformations in a body stresses in it are equal to zero.* It is clear from the above discussion that there must be no constant terms in equations (3.28) in this case, i.e., they must be homogeneous, as has been assumed at the outset.

Having established the form of relations (3.28), we can develop the expression for the potential elastic energy W. Comparing relations (3.27) and (3.28), we conclude that the partial derivatives of the function W represent linear homogeneous functions of the strain components

$$e_{xx},\ e_{yy},\ \ldots,\ e_{xy};$$

thus, the function W itself will be a homogeneous function of the second degree in these arguments. It can be obtained by integrating equations (3.27), but it is much easier to make use of Euler's known theorem on homogeneous functions which states that if $F(x, y, z, \ldots)$ is a homogeneous function of degree n, then

$$nF(x,\ y,\ z,\ \ldots)=\frac{\partial F}{\partial x}x+\frac{\partial F}{\partial y}y+\frac{\partial F}{\partial z}z+\ \ldots\ .$$

Applying it to the function W, we find that

$$2W=\frac{\partial W}{\partial e_{xx}}e_{xx}+\frac{\partial W}{\partial e_{yy}}e_{yy}+\ \ldots\ +\frac{\partial W}{\partial e_{xy}}e_{xy},$$

or, using (3.27), we get

$$2W=X_x e_{xx}+\ \ldots\ . \tag{3.32}$$

This expression can further be handled in two ways:

1. To replace the stress components by their expressions in terms of the strain components according to (3.28); then we obtain the required expression of W in the form of a homogeneous function of the second degree in the strain components.

2. To solve system (3.28) with respect to

$$e_{xx},\ e_{yy},\ \ldots,\ e_{xy}$$

and substitute these expressions in the preceding equality; then W will be represented as a homogeneous function of the second degree in the stress components

$$X_x,\ Y_y,\ \ldots,\ X_y.$$

[1] In the case considered, for instance, the magnitude of stresses depends upon the size of the sectorial cut-out $ACBD$.

In the case of an isotropic body, we use relations (3.13) and inverse relations (3.9) instead of (3.28); then we obtain the following expressions for the potential elastic energy:

(1) as a function of the strain components

$$2W = \lambda\theta^2 + 2\mu(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + \mu(e_{yz}^2 + e_{zx}^2 + e_{xy}^2), \quad (3.33)$$

or

$$2W = (\lambda + 2\mu)(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + \\ + 2\lambda(e_{xx}e_{yy} + e_{yy}e_{zz} + e_{zz}e_{xx}) + \mu(e_{yz}^2 + e_{zx}^2 + e_{xy}^2);$$

(2) as a function of the components of stress

$$2W = \frac{1}{E}[X_x^2 + Y_y^2 + Z_z^2 - 2\sigma(X_xY_y + Y_yZ_z + Z_zX_x) + \\ + 2(1+\sigma)(Y_z^2 + Z_x^2 + X_y^2)]. \quad (3.34)$$

From (3.33) we draw a very important conclusion that always

$$W > 0,$$

since $\lambda > 0$ and $\mu > 0$, as has been shown at the end of Sec. 18.

*22. ELASTIC CONSTANTS; REDUCTION IN THEIR NUMBER DUE TO THE EXISTENCE OF THE POTENTIAL OF ELASTIC FORCES

Equations (3.28) contain 36 constant coefficients a_{mn} called the elastic constants. They characterise the elastic properties of a body and by their dimension they are quite analogous to the moduli of elasticity E and G. The number of elastic constants of a body in general, as we see, is very great; however, it is reduced considerably when the potential of elastic forces exists.

Indeed, take a partial derivative of the potential energy with respect to any of the strains, for example, with respect to e_{yy}; on the basis of equations (3.27) and (3.28) we have

$$\frac{\partial W}{\partial e_{yy}} = Y_y = a_{21}e_{xx} + a_{22}e_{yy} + a_{23}e_{zz} + a_{24}e_{yz} + a_{25}e_{zx} + a_{26}e_{xy};$$

taking a partial derivative of both members once more, for example, with respect to e_{zx}, we get

$$\frac{\partial^2 W}{\partial e_{yy}\,\partial e_{zx}} = a_{25} \quad (3.35)$$

Now find the same second derivative by differentiating in reverse order

$$\left.\begin{aligned} \frac{\partial W}{\partial e_{zx}} &= Z_x, \\ \frac{\partial^2 W}{\partial e_{zx}\,\partial e_{yy}} &= \frac{\partial Z_x}{\partial e_{yy}} = a_{52}. \end{aligned}\right\} \qquad (3.36)$$

Since the magnitude of a derivative does not depend on the order of differentiation, we compare equations (3.35) and (3.36) and find $a_{25}=a_{52}$; the same can be proved, of course, for any two coefficients a_{mn} and a_{nm}

$$a_{mn}=a_{nm}. \qquad (3.37)$$

Relation (3.37) resulting from the existence of the potential of elastic forces shows that in equations (3.24) the coefficients which are symmetric in respect to the diagonal passing from the upper left corner to the lower right corner, are pairwise equal to each other. Consequently, in the most general case, 36 elastic constants will include 6 different constants

$$a_{11},\ a_{22},\ a_{33},\ a_{44},\ a_{55},\ a_{66},$$

which are located along the diagonal and $\frac{36-6}{2}=15$ among the remaining constants, making altogether $6+15=21$ constants.

Only "the most anisotropic" body exhibiting entirely different elastic properties in different directions can possess such a large number of elastic constants.

*23. ISOTROPIC BODY

Physical nonisotropic bodies (crystals) usually reveal greater or smaller symmetry of structure, as a result of which the number of elastic constants is considerably reduced; we shall consider here only the case of an isotropic body whose elastic properties are the same in all directions. For such a body equations (3.28) must not alter under any transformations of co-ordinates whatsoever. Hence we can easily reduce the number of elastic constants to 9 if we take into account the sign rule for shearing strains (Sec. 10); it follows from this rule that a shearing strain (e. g., e_{xy}) preserves its magnitude but changes the sign, if we reverse the direction of one of the axes in the plane of which the shearing strain occurs (Fig. 30).

Take, for instance, the first of equations (3.28) and reverse the direction of the axis Oy. This will obviously have no effect on

the left-hand member of the equation; the first three terms in the right-hand member (containing elongations) will remain unaltered; the fifth term will not change either; the fourth and sixth terms preserving the magnitude will change the sign; hence, the equality between the left-hand and right-hand members will be violated; this will not happen only when $a_{14}=0$, $a_{16}=0$, i.e., these two coefficients must vanish. If we reverse the direction of the axis Oz, we shall find in the same way that $a_{14}=a_{15}=0$; thus, we draw the general conclusion that $a_{14}=a_{15}=a_{16}=0$, i.e., *the normal stress X_x is not connected with shear deformations.* In view of the complete equality of the co-ordinate axes in an isotropic body this conclusion also applies to the other two normal

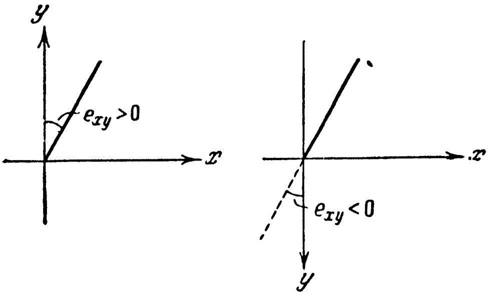

Fig. 30

stresses, Y_y and Z_z, i.e., $a_{24}=a_{25}=a_{26}=0$; $a_{34}=a_{35}=a_{36}=0$. On the basis of equation (3.37) we note now that the symmetric coefficients in the last three of equations (3.28) become zero:

$$\begin{gathered}a_{41}=a_{42}=a_{43}=0,\\ a_{51}=a_{52}=a_{53}=0,\\ a_{61}=a_{62}=a_{63}=0,\end{gathered}$$

i. e., *shearing stresses are not connected with elongations.*

The number of elastic constants has been reduced to 12 and equations (3.28) have fallen into two independent groups:

$$\begin{aligned}X_x&=a_{11}e_{xx}+a_{12}e_{yy}+a_{13}e_{zz}, & Y_z&=a_{44}e_{yz}+a_{45}e_{zx}+a_{46}e_{xy},\\ Y_y&=a_{21}e_{xx}+a_{22}e_{yy}+a_{23}e_{zz}, & Z_x&=a_{54}e_{yz}+a_{55}e_{zx}+a_{56}e_{xy},\\ Z_z&=a_{31}e_{xx}+a_{32}e_{yy}+a_{33}e_{zz}, & X_y&=a_{64}e_{yz}+a_{65}e_{zx}+a_{66}e_{xy}.\end{aligned}$$

Let us turn to the last three equations. If the direction of the axis Ox is reversed, the sign of the last two terms in the fourth equation will be changed to its opposite; the left-hand member of the equation will remain unaltered and, as before, we, therefore,

conclude that

$$a_{45}=a_{46}=0$$

and further

$$a_{54}=a_{56}=0; \quad a_{64}=a_{65}=0.$$

The number of elastic constants has been reduced to 12—3=9, as stated before,

$$\left.\begin{aligned} X_x &= a_{11}e_{xx}+a_{12}e_{yy}+a_{13}e_{zz},\\ Y_y &= a_{21}e_{xx}+a_{22}e_{yy}+a_{23}e_{zz},\\ Z_z &= a_{31}e_{xx}+a_{32}e_{yy}+a_{33}e_{zz},\\ Y_z &= a_{44}e_{yz},\\ Z_x &= a_{55}e_{zx},\\ X_y &= a_{66}e_{xy}. \end{aligned}\right\} \tag{3.38}$$

The transformations of co-ordinates which we have so far employed came to a rotation of particular axes through 180°; now we employ a rotation of axes through 90° by replacing Oy by Oz, Oz by Ox, Ox by Oy, etc. We take the first of equations (3.38) and interchange the axes Oy and Oz. This will have no effect on the left-hand member of the equation in view of the isotropy of the body; in the right-hand member, however, the elongations e_{yy} and e_{zz} will be interchanged; consequently, the equality will not be violated only when $a_{12}=a_{13}$; likewise, in the second and third equations we must have $a_{21}=a_{23}$, $a_{31}=a_{32}$ and finally on the basis of equality (3.37): $a_{12}=a_{13}=a_{23}=a_{21}=a_{31}=a_{32}$. Now successively replacing one axis by another and keeping in mind that the form of equations (3.38) must not be altered thereby, we come to the conclusion that

$$a_{11}=a_{22}=a_{33}; \quad a_{44}=a_{55}=a_{66};$$

hence, equations (3.38) become

$$\left.\begin{aligned} X_x &= a_{11}e_{xx}+a_{12}(e_{yy}+e_{zz}),\\ Y_y &= a_{11}e_{yy}+a_{12}(e_{zz}+e_{xx}),\\ Z_z &= a_{11}e_{zz}+a_{12}(e_{xx}+e_{yy}),\\ Y_z &= a_{44}e_{yz},\\ Z_x &= a_{44}e_{zx},\\ X_y &= a_{44}e_{xy}. \end{aligned}\right\} \tag{3.39}$$

Thus, the number of elastic constants has been reduced to three

$$a_{11}, \quad a_{12}, \quad a_{44}.$$

We have achieved this result by fulfilling the requirement that the form of the relations should not change with the rotation of the co-ordinate axes through 90° and 180°; however, this condition is not sufficient to guarantee the isotropy of a body, i.e., its complete homogeneity in all directions. Therefore, we make one more transformation of co-ordinates associated with the rotation of the axes through an arbitrary angle.

For simplicity we take the plane-strain situation when

$$e_{zz} = e_{yz} = e_{zx} = 0.$$

Then from equations (3.39) we have

$$\left.\begin{aligned} X_x &= a_{11}e_{xx} + a_{12}e_{yy}, \\ Y_y &= a_{11}e_{yy} + a_{12}e_{xx}, \\ X_y &= a_{44}e_{xy}. \end{aligned}\right\} \tag{3.40}$$

Consider the case of simple tension in the direction of the x axis when

$$\begin{aligned} Y_y &= a_{11}e_{yy} + a_{12}e_{xx} = 0, \\ X_y &= a_{44}e_{xy} = 0. \end{aligned}$$

Hence

$$e_{xy} = 0; \quad e_{yy} = -\frac{a_{12}}{a_{11}} e_{xx} \tag{3.41}$$

and the first of equations (3.40) yields

$$X_x = \frac{a_{11}^2 - a_{14}^2}{a_{11}} e_{xx}.$$

Next we calculate the shearing stress U_v on an arbitrary area inclined to the plane Oxz at an angle α; this can be done in two ways. On the one hand, according to the second of formulas (1.15) which gives

$$U_v = X_x \sin\alpha\cos\alpha = \frac{X_x}{2}\sin 2\alpha = \frac{a_{11}^2 - a_{12}^2}{2a_{11}} e_{xx}\sin 2\alpha.$$

On the other hand, use can be made of the last of equations (3.40), since the body is assumed to be isotropic and this equation is therefore valid for any direction of the co-ordinate axes:

$$U_v = a_{44}e_{uv}.$$

Comparing both expressions of U_v, we get

$$(a_{11}^2 - a_{12}^2)\, e_{xx}\sin 2\alpha = 2a_{11}a_{44}e_{uv}. \tag{3.42}$$

We now establish a relation between the strain components e_{xx} and e_{uv} which appear in (3.42). For this purpose (Fig. 31) we take a square $ABCD$ with the side equal to unity in the plane Oxy. It is known from experiment that in the case of simple extension in the direction of the axis Ox horizontal segments elongate while vertical segments shorten and the square takes the shape $A_1B_1C_1D_1$. Let us calculate the angles of rotation of the sides of the square

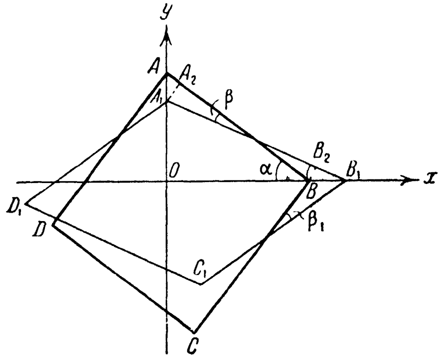

Fig. 31

and then the shear angle e_{uv} as the distortion of the right angle ABC. We have

$$OA = 1 \cdot \sin\alpha; \quad OB = 1 \cdot \cos\alpha,$$

and

$$AA_1 = 1 \cdot \sin\alpha\,(-e_{yy}), \quad BB_1 = 1 \cdot \cos\alpha \cdot e_{xx}.$$

Assuming the deformation to be small we obtain, to small quantities of the second order,

$$A_1A_2 = AA_1 \cos\alpha = 1 \cdot \sin\alpha\cos\alpha\,(-e_{yy}),$$

$$BB_2 = BB_1 \sin\alpha = 1 \cdot \sin\alpha\cos\alpha \cdot e_{xx}.$$

We calculate the angle of rotation β of the side AB with the same degree of accuracy

$$\beta = \frac{A_1A_2 + BB_2}{AB} = 1\,(e_{xx} - e_{yy})\sin\alpha\cos\alpha,$$

or

$$\beta = 1\,\frac{e_{xx} - e_{yy}}{2}\sin 2\alpha.$$

The angle of rotation β_1 of the other face BC will be obtained from the same formula by replacing α by $\alpha - \frac{\pi}{2}$.

$$\beta_1 = -1\,\frac{e_{xx} - e_{yy}}{2}\sin 2\alpha.$$

The required shear angle is

$$e_{uv} = \beta - \beta_1 = (e_{xx} - e_{yy})\sin 2\alpha,$$

or on the basis of (3.41)

$$e_{uv} = \frac{a_{11} + a_{12}}{a_{11}}\, e_{xx}\sin 2\alpha.$$

This is the desired relation between e_{xx} and e_{uv}; substituting this value of e_{uv} in (3.42) we find after a simple transformation

$$a_{11} - a_{12} = 2a_{44}. \tag{3.43}$$

Thus, the three elastic constants are governed by the linear relation (3.43) indicating that only two of them are independent; the third constant will be expressed through them by means of equation (3.43). Taking a_{12} and a_{44} as independent constants, we find

$$a_{11} = a_{12} + 2a_{44}.$$

Consequently, an isotropic elastic body is characterised by only two elastic constants: a_{12} and a_{44}. These are Lamé's coefficients that we denoted by λ and μ in Sec. 18. Returning to this notation, $a_{12} = \lambda$, $a_{44} = \mu$, $a_{11} = \lambda + 2\mu$, and substituting the same in equations (3.39), we immediately reduce these equations to the form (3.13) given in Sec. 18. Hence, we come to the conclusion that equations (3.13) follow directly from the most general equations (3.28), if one accepts the hypothesis of the existence of the potential of elastic forces and if a given body is assumed to be isotropic.

In Sec. 16 we presented without proof the relation among three elastic constants E, μ, G:

$$G = \frac{E}{2(1+\sigma)}.$$

It can now be shown that this relation is obtainable as a consequence of the generalised Hooke's law (3.13), containing *only two* elastic constants λ and μ:

$$\left.\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx}; & Y_z &= \mu e_{yz}; \\ Y_y &= \lambda\theta + 2\mu e_{yy}; & Z_x &= \mu e_{zx}; \\ Z_z &= \lambda\theta + 2\mu e_{zz}; & X_y &= \mu e_{xy} \end{aligned}\right\} \tag{3.44}$$

By adding up the first three equalities, we obtained Hooke's law in the volumetric form in Sec. 18:

$$\Theta = (3\lambda + 2\mu)\,\theta \quad \text{or} \quad \theta = \frac{\Theta}{3\lambda + 2\mu}.$$

Substituting this value of θ in the first of equalities (3.44), we find the expression of e_{xx} in terms of stresses

$$e_{xx} = \frac{X_x}{2\mu} - \frac{\lambda}{2\mu\,(3\lambda + 2\mu)}\,\Theta;$$

let us compare this expression with the first of equalities (3.6) written down in the form

$$e_{xx} = \frac{1+\sigma}{E} X_x - \frac{\sigma}{E}\,\Theta.$$

Their left-hand members are identical; by comparing the right-hand members, we obtain

$$\mu = G = \frac{E}{2\,(1+\sigma)}; \qquad \frac{\lambda}{2\mu\,(3\lambda + 2\mu)} = \frac{\sigma}{E}. \tag{3.45}$$

We have taken into account here that from comparison of the last of equalities (3.9) and (3.13) we get $\mu = G$; the first of equalities (3.45) gives the required relation (3.4); this together with the second equality yields, as in (3.16),

$$\sigma = \frac{\lambda}{2\,(\lambda + \mu)}.$$

IV

Solution of the Elasticity Problem in Terms of Displacements

24. COMPENDIUM OF BASIC EQUATIONS OF THE THEORY OF ELASTICITY

We have completed the derivation of all basic groups of equations of the theory of elasticity with the generalised Hooke's law; let us present them once more in the general list and for convenience in making references in future furnish them with special numeration (R-an numerals).

A. Static equations

1. Differential conditions of equilibrium (Navier's equations) (1.5):

$$\left.\begin{aligned}
&\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+\rho X=0\left(=\rho\frac{\partial^2 u}{\partial t^2}\right);\\
&\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+\rho Y=0\left(=\rho\frac{\partial^2 v}{\partial t^2}\right);\\
&\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+\rho Z=0\left(=\rho\frac{\partial^2 w}{\partial t^2}\right).
\end{aligned}\right\} \tag{I}$$

2. Surface conditions (1.8):

$$\left.\begin{aligned}
X_v&=X_x\cos(vx)+X_y\cos(vy)+X_z\cos(vz);\\
Y_v&=Y_x\cos(vx)+Y_y\cos(vy)+Y_z\cos(vz);\\
Z_v&=Z_x\cos(vx)+Z_y\cos(vy)+Z_z\cos(vz).
\end{aligned}\right\} \tag{II}$$

B. *Geometrical equations*

3. Relation between displacements and strains (Cauchy's equations) (2.6):

$$\left.\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x}; & e_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; \\ e_{yy} &= \frac{\partial v}{\partial y}; & e_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \\ e_{zz} &= \frac{\partial w}{\partial z}; & e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \end{aligned}\right\} \tag{III}$$

4. Compatibility equations (Saint-Venant's equations) (2.15):

$$\left.\begin{aligned} &\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x\, \partial y}; & &\frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} - \frac{\partial e_{xy}}{\partial z}\right) = 2\frac{\partial^2 e_{zz}}{\partial x\, \partial y}; \\ &\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{\partial^2 e_{yz}}{\partial y\, \partial z}; & &\frac{\partial}{\partial x}\left(\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} - \frac{\partial e_{yz}}{\partial x}\right) = 2\frac{\partial^2 e_{xx}}{\partial y\, \partial z}; \\ &\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = \frac{\partial^2 e_{zx}}{\partial z\, \partial x}; & &\frac{\partial}{\partial y}\left(\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{zx}}{\partial y}\right) = 2\frac{\partial^2 e_{yy}}{\partial z\, \partial x}. \end{aligned}\right\} \tag{IV}$$

C. *Physical equations*

5. Generalised Hooke's law (3.9), (3.13):

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E}[X_x - \sigma(Y_y + Z_z)]; & e_{yz} &= \frac{1}{G}Y_z; \\ e_{yy} &= \frac{1}{E}[Y_y - \sigma(Z_z + X_x)]; & e_{zx} &= \frac{1}{G}Z_x; \\ e_{zz} &= \frac{1}{E}[Z_z - \sigma(X_x + Y_y)]; & e_{xy} &= \frac{1}{G}X_y. \end{aligned}\right\} \tag{V}$$

$$\theta = \frac{1-2\sigma}{E}\Theta. \tag{Va}$$

$$\left.\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx}; & Y_z &= \mu e_{yz}; \\ Y_y &= \lambda\theta + 2\mu e_{yy}; & Z_x &= \mu e_{zx}; \\ Z_z &= \lambda\theta + 2\mu e_{zz}; & X_y &= \mu e_{xy}. \end{aligned}\right\} \tag{V$'$}$$

$$\Theta = (3\lambda + 2\mu)\,\theta. \tag{V$'$a}$$

On the basis of these groups of equations it is possible to proceed directly to the solution of the general problem of the theory of elasticity for the stresses and strains produced in an isotropic elastic body under the action of external forces.

It should be noted once again that all the foregoing equations have been derived on the assumption of very small deformations;

this was already mentioned in Sec. 10 relative to equations (III) and, consequently, to (IV); equations (I) have been developed for the unstrained state of a body and may be considered true only in the case of very small deformations.

Equations (I)-(V) involve a large number of unknown functions representing stresses, displacements and strains. It is, therefore, necessary first to choose those quantities which we shall take as basic unknowns to be determined in the first instance and with which it will subsequently be possible to find all the remaining factors characterising the state of stress and deformation of a body.

The history of the development of the theory of elasticity shows that here one can proceed in two different ways.

(1) To take displacements of points in an elastic body as basic unknowns; then we have three unknowns at each point (x, y, z) of the body:

$$u(x, y, z), \quad v(x, y, z), \quad w(x, y, z), \tag{4.1}$$

and the problem will be reduced to the determination of three functions (4.1) with three conditions of equilibrium (I) to be satisfied; at the same time it will be necessary to satisfy conditions (II) on the surface which contain external forces (load)

$$X_v, \quad Y_v, \quad Z_v.$$

(2) To take stresses as basic unknowns; then we have six unknowns at each point of a body

$$\left.\begin{aligned} X_x(x, y, z), \quad Y_y(x, y, z), \quad Z_z(x, y, z), \\ Y_z(x, y, z), \quad Z_x(x, y, z), \quad X_y(x, y, z). \end{aligned}\right\} \tag{4.2}$$

The problem is reduced to the determination of six functions (4.2); they must satisfy three equations of equilibrium (I); these equations, however, are inadequate, and it will be necessary to make appeal to six compatibility conditions (IV). At the same time conditions (II) must be satisfied on the surface, just as in the first method.

It appears from these arguments that the first procedure ("the solution of the problem in terms of displacements") is simpler from a mathematical point of view, for it involves a smaller number of unknowns and one has to deal with a smaller number of equations. After determining displacements (4.1), we readily find strains $e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy}$ from equations (III) and, finally, substituting them in the equations of Hooke's law (V′), we obtain stresses

$$X_x, \ Y_y, \ Z_z. \ Y_z, \ Z_x, \ X_y.$$

In the present chapter we shall consider the method of solving the problem of the theory of elasticity in terms of displacements and apply it to several particular cases.

25. LAMÉ'S EQUATIONS

Let us write down the differential equations of equilibrium (I)

$$\left.\begin{aligned}
\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+X\rho &= 0\left(=\rho\frac{\partial^2 u}{\partial t^2}\right),\\
\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+Y\rho &= 0\left(=\rho\frac{\partial^2 v}{\partial t^2}\right),\\
\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+Z\rho &= 0\left(=\rho\frac{\partial^2 w}{\partial t^2}\right),
\end{aligned}\right\} \tag{I}$$

and the conditions on the surface (II)

$$\left.\begin{aligned}
X_v &= X_x\cos(vx)+X_y\cos(vy)+X_z\cos(vz);\\
Y_v &= Y_x\cos(vx)+Y_y\cos(vy)+Y_z\cos(vz);\\
Z_v &= Z_x\cos(vx)+Z_y\cos(vy)+Z_z\cos(vz).
\end{aligned}\right\} \tag{II}$$

All these equations have to be transformed to express stresses in terms of displacements (4.1) which we take as unknowns. For this purpose we express stresses in terms of strains according to Hooke's law (V′), and the strains in terms of displacements according to (III). Selecting in (V′) the stresses entering in the first of equations (I), we get

$$\left.\begin{aligned}
X_x &= \lambda\theta+2\mu\frac{\partial u}{\partial x},\\
X_y &= \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right),\\
X_z &= \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right).
\end{aligned}\right\} \tag{4.3}$$

By differentiating them, we obtain

$$\left.\begin{aligned}
\frac{\partial X_x}{\partial x} &= \lambda\frac{\partial\theta}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial x^2},\\
\frac{\partial X_y}{\partial y} &= \mu\frac{\partial^2 v}{\partial x\,\partial y}+\mu\frac{\partial^2 u}{\partial y^2},\\
\frac{\partial X_z}{\partial z} &= \mu\frac{\partial^2 w}{\partial x\,\partial z}+\mu\frac{\partial^2 u}{\partial z^2}.
\end{aligned}\right\}$$

Substituting this in the first of equations (I), we get

$$\lambda\frac{\partial\theta}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}\right)+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+X\rho=$$
$$=0\left(+\rho\frac{\partial^2 u}{\partial t^2}\right). \quad (4.4)$$

We note that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\frac{\partial\theta}{\partial x},$$

and introduce a compact notation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\nabla^2 u.$$

This expression is called Laplacian operator or Laplace's operator on the function $u(x, y, z)$. Hence, equation (4.4) becomes

$$(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\nabla^2 u+X\rho=0\left(=\rho\frac{\partial^2 u}{\partial t^2}\right). \quad (4.5)$$

In a similar manner we transform the other two of equations (I); they are obtained directly from (4.5) by a cyclic change between the letters (x, y, z) and (u, v, w); we arrive at the following *system of basic equations of the theory of elasticity for the determination of displacements*:

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\nabla^2 u+\rho X&=0\left(=\rho\frac{\partial^2 u}{\partial t^2}\right);\\(\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\nabla^2 v+\rho Y&=0\left(=\rho\frac{\partial^2 v}{\partial t^2}\right);\\(\lambda+\mu)\frac{\partial\theta}{\partial z}+\mu\nabla^2 w+\rho Z&=0\left(=\rho\frac{\partial^2 w}{\partial t^2}\right).\end{aligned}\right\} \quad \text{(VI)}$$

If the body forces ρX, ρY, ρZ are absent, equations (VI) will be homogeneous. In the case of equilibrium, their right-hand members are equal to zero; dividing all the equations by μ and introducing the elastic constants μ and k (3.17), we reduce them to

$$\left.\begin{aligned}\nabla^2 u+(k+1)\frac{\partial\theta}{\partial x}&=0;\\\nabla^2 v+(k+1)\frac{\partial\theta}{\partial y}&=0;\\\nabla^2 w+(k+1)\frac{\partial\theta}{\partial z}&=0.\end{aligned}\right\} \quad \text{(VI}'\text{)}$$

Equations (VI) are known as Lamé's equations. They involve a synthesis of the theories of stresses, deformations and the relation between them—the theories elucidated in the preceding three chapters. Consequently, Lamé's equations include all the premises

of a mechanical, geometrical and purely physical character on which the theory of elasticity is based. In fact,

(1) they express the conditions of equilibrium of each element of a body (if their right-hand members are equal to zero) or represent the equations of motion of this element;

(2) they contain the geometrical characteristics of deformations u, v, w and θ;

(3) they contain the physical factors λ, μ and ρ characterising the elastic properties and density of a body.

From these considerations alone we can estimate a priori the enormous role equations (VI) play in the theory of elasticity by making it possible to approach the solution of a number of very important problems.

In the same way we transform the surface conditions (II); replacing in the first of them the stresses by their expressions (4.3), we have

$$X_v = \lambda\theta\cos(vx) + \mu\left[\frac{\partial u}{\partial x}\cos(vx) + \frac{\partial u}{\partial y}\cos(vy) + \frac{\partial u}{\partial z}\cos(vz)\right] + \\ + \mu\left[\frac{\partial u}{\partial x}\cos(vx) + \frac{\partial v}{\partial x}\cos(vy) + \frac{\partial w}{\partial x}\cos(vz)\right]. \qquad (4.6)$$

The right-hand member of this equation can be written in a simpler form if we bear in mind that the first bracketed expression represents the derivative of the function $u(x, y, z)$ along the normal to the surface of the body:

$$\frac{\partial u}{\partial V} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial V} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial V} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial V} = \frac{\partial u}{\partial x}\cos(vx) + \\ + \frac{\partial u}{\partial y}\cos(vy) + \frac{\partial u}{\partial z}\cos(vz).$$

Reasoning in the same way with regard to the other two equations (II), we obtain the definitive form of the conditions on the surface:

$$\left.\begin{aligned} X_v &= \lambda\theta\cos(vx) + \mu\frac{\partial u}{\partial V} + \\ &\quad + \mu\left[\frac{\partial u}{\partial x}\cos(vx) + \frac{\partial v}{\partial x}\cos(vy) + \frac{\partial w}{\partial x}\cos(vz)\right]; \\ Y_v &= \lambda\theta\cos(vy) + \mu\frac{\partial v}{\partial V} + \\ &\quad + \mu\left[\frac{\partial u}{\partial y}\cos(vx) + \frac{\partial v}{\partial y}\cos(vy) + \frac{\partial w}{\partial y}\cos(vz)\right]; \\ Z_v &= \lambda\theta\cos(vz) + \mu\frac{\partial w}{\partial V} + \\ &\quad + \mu\left[\frac{\partial u}{\partial z}\cos(vx) + \frac{\partial v}{\partial z}\cos(vy) + \frac{\partial w}{\partial z}\cos(vz)\right]. \end{aligned}\right\} \qquad \text{(VIa)}$$

In conjunction with the surface conditions (VIa), Lamé's equations (VI) permit us to proceed directly to the solution of problems of the theory of elasticity. If we are in a position to integrate equations (VI) and find the functions u, v and w which satisfy the conditions on the surface in the form (II) or (VIa), then, by introducing them in equations (III), we determine strains $e_{xx}, \ldots,$ e_{xy}; by introducing the latter in the equations of Hooke's law (V′), we obtain the stresses

$$X_x, Y_y, \ldots, X_y.$$

Integration of equations (VI), subject to the surface conditions (VIa), is a very difficult task. In solving many problems of practical importance, however, it is convenient to employ the inverse method by prescribing displacements as functions of the co-ordinates of a point (x, y, z) and seeking, on the basis of conditions (VIa), the external forces on the surface of a body (load) to which the given displacements correspond. Saint-Venant's "semi-inverse" method has proved to be very fruitful too; according to this method, it is necessary to prescribe a part of external forces and a part of displacements, while the remaining factors are sought from the condition that the basic equations (VI) and (VIa) are satisfied. By way of illustration, we shall apply both methods to problems encountered in practice.

26. LONGITUDINAL AND TRANSVERSE VIBRATIONS IN AN UNBOUNDED ELASTIC MEDIUM

Let us apply the inverse method to this problem, i.e., prescribe displacements and verify whether they are possible in a homogeneous elastic medium, in other words, whether they satisfy Lamé's equations (VI). Since we are studying the case of motion, displacements (4.1) (Sec. 17) should depend not only on the co-ordinates of a point, but also on the time t.

A. Let us choose the following expressions of displacements:

$$\left.\begin{aligned} u &= u(x, t), \\ v &= 0, \\ w &= 0, \end{aligned}\right\} \qquad (4.7)$$

and, moreover, suppose that the body forces are absent, i.e.,

$$X = Y = Z = 0.$$

Since $v = w = 0$, the displacements of all points take place parallel to the axis Ox; furthermore, the displacement u does not depend on

y, z; consequently, if we (Fig. 32) consider points which in the absence of motion are located in the plane P normal to the axis Ox, all these points will be displaced equally and simultaneously; in other words, the plane P will move in the direction of the axis Ox without deforming.

Indeed, the equation of the plane P at rest will be $x=x_0$. It will have the form $x=x_0+u$ or $x=x_0+u(x_0, t)$ at any instant during the motion. This is again a plane parallel to the plane yOz but located at a time-dependent variable distance from the plane yOz. If we select a number of such planes P, P_1, P_2, ... in our elastic medium, they will all displace normally to Ox, coming together or moving away from each other [1]. In this case the motion described by equations (4.7) is called uniform longitudinal vibration along the axis Ox.

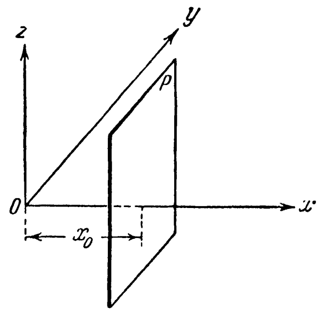

Fig. 32

The surface conditions (VIa) need not be satisfied, since the medium has been assumed to be unbounded.

In order to check the possibility of vibrations governed by equations (4.7) we substitute them in equations (VI). First we calculate

$$\theta=\frac{\partial u}{\partial x}; \quad \frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial x^2}; \quad \frac{\partial\theta}{\partial y}=\frac{\partial^2 u}{\partial x\,\partial y}=0; \quad \frac{\partial\theta}{\partial z}=\frac{\partial^2 u}{\partial x\,\partial z}=0;$$

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}; \quad \nabla^2 v=0; \quad \nabla^2 w=0; \quad \frac{\partial^2 v}{\partial t^2}=\frac{\partial^2 w}{\partial t^2}=0.$$

We now observe that the second and third of equations (VI) are satisfied identically; the first equation is transformed as

$$(\lambda+\mu)\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial x^2}=\rho\frac{\partial^2 u}{\partial t^2},$$

or

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, \tag{4.8}$$

where

$$a^2=\frac{\lambda+2\mu}{\rho}. \tag{4.9}$$

---

[1] Since the distance d between any two planes $x=x_0$ and $x=x_1$ depends on the time t during the motion,

$$d=x_1+u(x_1, t)-[x_0+u(x_0, t)]=x_1-x_0+u(x_1, t)-u(x_0, t).$$

Hence we draw the basic conclusion: the longitudinal vibratory motion governed by equations (4.7) *is possible if the function* $u(x, t)$ *satisfies differential equation* (4.8).

B. Let us now choose the following expressions of displacements:

$$\left.\begin{aligned} u &= 0, \\ v &= 0, \\ w &= w(x, t). \end{aligned}\right\} \qquad (4.10)$$

Let the body forces be absent as before:

$$X = Y = Z = 0.$$

In this case all displacements take place parallel to the axis Oz. By reasoning as above, we easily deduce that all points of any plane P (Fig. 32) displace equally and simultaneously, remaining at a constant distance from the plane zOy. If several such parallel planes are considered, they will move in a vertical direction. In the case of a periodic motion we shall have *uniform transverse vibration* along the axis Oz. Let us check the possibility of such vibrations; proceeding from equations (4.10), we have

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0;$$

thus, there is no dilatation. Furthermore,

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial z} = 0; \qquad \nabla^2 u = \nabla^2 v = 0;$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2}; \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} = 0.$$

In these conditions, the first and second of equations (VI) are satisfied identically; the third equation becomes

$$\mu \frac{\partial^2 w}{\partial x^2} = \rho \frac{\partial^2 w}{\partial t^2}$$

or

$$\frac{\partial^2 w}{\partial t^2} = b^2 \frac{\partial^2 w}{\partial x^2}, \qquad (4.11)$$

where

$$b^2 = \frac{\mu}{\rho}. \qquad (4.12)$$

Vibration (4.10) *is thus possible if the function* $w(x, t)$ *satisfies differential equation* (4.11). We observe that equations (4.8) and (4.11) are alike in form and differ only by the value of the constant factor [cf. formulas (4.9) and (4.12)].

C. Consider the particular case of harmonic vibrations. Let us

[cf. equations (4.7)]:

$$u = A \sin 2\pi \left(\frac{x}{l} - \frac{t}{T}\right). \tag{4.13}$$

Substituting this expression in equation (4.8) and cancelling, we obtain

$$\frac{1}{T^2} = \frac{a^2}{l^2},$$

or

$$\frac{l}{T} = \pm a = \pm \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{4.14}$$

Consequently, vibration (4.13) is possible in an unbounded elastic medium if the parameters l and T satisfy relationship (4.14).

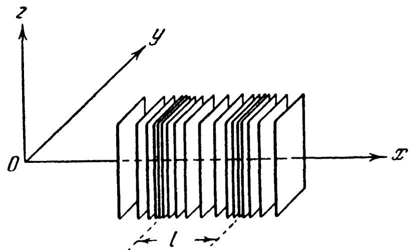

Fig. 33

The parameter A (the amplitude of vibration) remains arbitrary. The parameter T is the period of vibration; in fact, if x is held constant (i.e., if we consider one and the same plane P in Fig. 32 at all times) and the time t is given an increment T, the deviation u will not change, according to equation (4.13).

The parameter l is the wave length. Its geometrical meaning can best be clarified by considering the unit elongation

$$e_{xx} = \frac{\partial u}{\partial x} = A \frac{2\pi}{l} \cos 2\pi \left(\frac{x}{l} - \frac{t}{T}\right). \tag{4.15}$$

Let t be kept constant, i.e., let us consider all the possible planes P (Fig. 32) at a certain instant of time. As e_{xx} is represented by a periodic function of x, formula (4.15) shows that by giving the abscissa x an increment l, we shall obtain a new plane P at whose points e_{xx} has the same value; let us consider, for example, min e_{xx}; this will be the case at those places where the thickness of the planes P is greatest at the given instant of time t (Fig. 33); the distance between these points, as we have pointed out, is equal to l, and accordingly l is called the wave length. We shall obtain the magnitude of the maximum contrac-

tion (extension), according to equation (4.15), if we put

$$\cos 2\pi\left(\frac{x}{l}-\frac{t}{T}\right)=\pm 1$$

or

$$\frac{x}{l}-\frac{t}{T}=\frac{m}{2}, \tag{4.16}$$

where m is an integer.

Equation (4.16) shows that the abscissa x of this point moves uniformly as time passes; the velocity of this motion is

$$V=\frac{dx}{dt}=\frac{l}{T}.$$

This is the velocity of wave propagation. On the basis of equation (4.14) we have

$$V=\pm a=\pm\sqrt{\frac{\lambda+2\mu}{\rho}}. \tag{4.17}$$

We conclude from the foregoing analysis that the harmonic vibration given by (4.13) is possible in a homogeneous elastic medium. The amplitude A is arbitrary; the wave length l and the period of vibration T may also be variable, but *the velocity of propagation of longitudinal wave* $V=\frac{l}{T}$, *according to equation* (4.17), *is a constant which depends on the elastic coefficients* λ *and* μ *and the density* ρ *of a medium.*

The longitudinal vibrations under consideration are the so-called sonic vibrations, since for

$$\frac{1}{16}\text{ sec}>T>\frac{1}{50{,}000}\text{ sec}$$

these vibrations are heard as a sound. Formula (4.17) is important because it gives the velocity of propagation of sound in a solid of infinite dimensions. Since formula (4.17) simultaneously gives the velocity of propagation of longitudinal strain e_{xx} [cf. formula (4.15)], it follows that longitudinal strains are propagated through an elastic medium with a quite definite velocity equal to the velocity of sound.

27. GENERAL SOLUTION OF THE EQUATION OF VIBRATIONS

We have considered only the case of harmonic longitudinal vibration according to law (4.13); however, there can be various types of vibrations in an elastic body. Indeed, the most general

solution of equation (4.8) is

$$u = \varphi(x - at) + \psi(x + at), \tag{4.18}$$

where φ and ψ are arbitrary functions; substituting this expression in equation (4.8), we see that it is satisfied identically. It can easily be shown that the right-hand member of equation (4.18) represents two vibrations travelling with the velocity a in the positive and negative directions of the axis Ox.

Take, for example, the particular solution

$$u = \varphi(x - at). \tag{4.19}$$

If the independent variables x and t are connected by the condition

$$x - at = C \tag{4.20}$$

[cf. equality (4.16)], we obtain: $u = \varphi(C) = \text{constant}$. Equation (4.20) indicates that a point with a given constant value of the deviation u moves uniformly with the velocity

$$\frac{dx}{dt} = V = a.$$

It can be seen that equation (4.13) is a particular case of equation (4.19); in fact, it may be rewritten as

$$u = A \sin \frac{2\pi}{l}(x - at), \quad \text{where} \quad a = \frac{l}{T}.$$

Similarly, the particular solution $u = \psi(x + at)$ corresponds to vibration travelling with the same velocity in the negative direction of the axis Ox.

The ultimate conclusion: the velocity of propagation of longitudinal vibration in a homogeneous elastic body is constant and is given by formula (4.17). The law of vibration may be arbitrary on account of the arbitrariness of functions φ and ψ in the general solution (4.18) of equation (4.8).

These considerations are valid, of course, with respect to equation (4.11) of transverse vibrations as well.

Its general solution is

$$w = \varphi(x - bt) + \psi(x + bt).$$

Here b is the velocity of propagation of transverse vibrations. Consider a special case of harmonic transverse vibration

$$w = B \sin 2\pi\left(\frac{x}{l_1} - \frac{t}{T_1}\right) = B \sin \frac{2\pi}{l_1}(x - bt),$$

where

$$b = \frac{l_1}{T_1}. \tag{4.21}$$

Here l_1 is the wave length of transverse vibrations (Fig. 34); T_1 is the period of vibration. As in the case of longitudinal vibrations, we find that the velocity of propagation of vibration [cf. formulas (4.21) and (4.12)] is

$$V_1 = \frac{l_1}{T_1} = b = \pm\sqrt{\frac{\mu}{\rho}}. \tag{4.22}$$

Comparing (4.17) and (4.22), we see that

$$|V_1| < |V|, \tag{4.23}$$

i.e., the velocity of propagation of transverse vibrations is less than that of longitudinal vibrations. Their ratio is

$$\left|\frac{V_1}{V}\right| = \sqrt{\frac{\mu}{\lambda+2\mu}} = \sqrt{\frac{1-2\sigma}{2-2\sigma}}; \tag{4.24}$$

it depends only on the elastic constants of a medium.

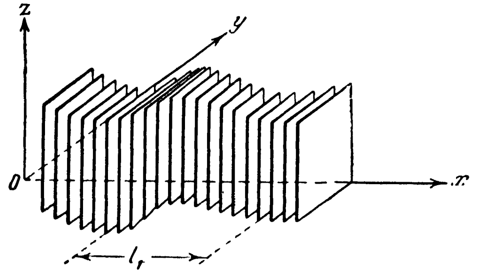

Fig. 34

On the basis of equations (III) of Sec. 24 and (4.10) we conclude that transverse vibrations are accompanied by shear

$$e_{xz} = \frac{\partial w}{\partial x}.$$

This shear is propagated through an elastic medium with the velocity given by formula (4.22). From formulas (4.23) and (4.24) it follows that transverse strains (shearing strains) are propagated through an isotropic elastic medium far more slowly than longitudinal strains. Indeed, if we set, for example, $\sigma = \frac{1}{3}$ in (4.24), we obtain

$$\left|\frac{V_1}{V}\right| = \sqrt{\frac{1-\frac{2}{3}}{2-\frac{2}{3}}} = \frac{1}{2}.$$

The velocities V and V_1 themselves, as is known, are very great; let us calculate, for instance, the velocity of propagation of longitudinal vibrations for steel; according to (4.14), we have

$$V = a = \sqrt{\mu \frac{\frac{\lambda}{\mu} + 2}{\rho}} = \sqrt{\mu \frac{\frac{2\sigma}{1-2\sigma} + 2}{\rho}} =$$

$$= \sqrt{\frac{\mu}{\rho} \frac{2(1-\sigma)}{1-2\sigma}} = \sqrt{\frac{E}{\rho} \frac{1-\sigma}{(1+\sigma)(1-2\sigma)}}.$$

For steel $E = 2 \times 10^6$ kg/cm², $\rho = \frac{7.85}{10^3 \times 981}$ kg/cm³, $\sigma = 0.3$; substituting this in the preceding formula, we get

$$V = a = \sqrt{\frac{2 \times 10^9 \times 981\,(1-0.3)}{7.85\,(1+0.3)\,(1-0.6)}} = 10^4 \sqrt{3{,}350} =$$

$$= 580{,}000 \text{ cm/sec} = 5{,}800 \text{ m/sec}.$$

It should be noted, however, that with such a great velocity of elastic waves involved, the velocities of the particles of the body $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$ remain very small.

28. LONGITUDINAL VIBRATIONS OF A BAR. FOURIER'S METHOD

The preceding sections dealt with an unbounded elastic medium, i.e., with a medium extending unlimitedly in the direction of

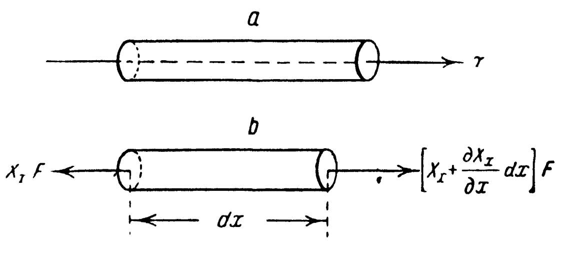

Fig. 35

the three co-ordinate axes. If we consider a thin cylindrical bar (Fig. 35 a), its vibrations will occur in a manner somewhat different from that for an unbounded medium; for instance, when subjected to longitudinal vibrations, its lateral dimensions will vary along with longitudinal elongations e_{xx}; consequently, in addition to primary displacements u along the axis Ox there will

appear secondary transverse displacements v and w, and the phenomenon will become much more complicated. When it is a question of a thin bar, however, transverse strains may be neglected without introducing serious error, and that will make it easy to derive the differential equation of longitudinal vibrations of such a bar.

Let us isolate (Fig. 35 *b*) from the bar an element of length dx and replace the action of the removed parts by the forces

$$-X_x F \text{ and } \left(X_x + \frac{\partial X_x}{\partial x} dx\right) F,$$

where F is the cross-sectional area of the bar. The mass of the isolated element is $\rho F\, dx$ and the equation of its motion is

$$\rho F\, dx \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x} dx F, \tag{4.25}$$

or

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x}.$$[1]

Since we assume that the bar is subjected to simple extension or compression only, in accordance with Hooke's law, we have

$$X_x = E e_{xx} = E \frac{\partial u}{\partial x},$$

and, consequently,

$$\frac{\partial X_x}{\partial x} = E \frac{\partial^2 u}{\partial x^2}.$$

Substituting this expression in equation (4.25), we get

$$\frac{\partial^2 u}{\partial t^2} = a_1^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.26}$$

where

$$a_1^2 = \frac{E}{\rho}. \tag{4.27}$$

Equation (4.26) differs from equation (4.8) for an unbounded medium only by the constant factor [cf. formulas (4.9) and (4.27)]; the difference is due to the fact that in the case of an unbounded medium we assumed that there were no transverse displacements v and w and respective strains, and this, in turn, sets up transverse stresses Y_y and Z_z; in the case of the bar, however, these stresses do not occur. Equation (4.27) makes it possible to calculate the modulus of elasticity of the bar E if its density ρ and the velocity of propagation of sound in it a_1 are known.

[1] This equality coincides, obviously, in the problem under consideration with the first of Navier's equations (I).

The general solution of the differential equation (4.26) has the previous form (4.18) [1], but in solving specific problems one most frequently uses integration of equation (4.26) in Fourier series. For this purpose we write the equation in the general form:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.28}$$

where α equals a, b, a_1 depending on the character of the problem being solved. Let us attempt to find a particular solution of equation (4.28) in the form

$$u = XT, \tag{4.29}$$

where X is a function of one variable x, T is a function of one variable t. Substituting this in equation (4.28), we find

$$X \frac{d^2T}{dt^2} = \alpha^2 T \frac{d^2X}{dx^2}$$

or, separating the variables,

$$\frac{1}{\alpha^2 T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2}. \tag{4.30}$$

The left-hand member of this equation depends only on t, while the right-hand member depends only on x; since both of these variables are independent, equation (4.30) can be satisfied identically only in the case if its left-hand and right-hand members are equal separately to one and the same constant number; designate it by $-\lambda^2$; then equation (4.30) is decomposed into two ordinary differential equations

$$\frac{d^2T}{dt^2} = -\lambda^2 \alpha^2 T; \quad \frac{d^2X}{dx^2} = -\lambda^2 X.$$

Their general solutions are

$$T = A \cos \lambda \alpha t + B \sin \lambda \alpha t,$$
$$X = C \cos \lambda x + D \sin \lambda x.$$

Substituting this in equation (4.29), we obtain a particular solution of equation (4.28):

$$u = (A \cos \lambda \alpha t + B \sin \lambda \alpha t)(C \cos \lambda x + D \sin \lambda x). \tag{4.31}$$

Varying A, B, C, D, λ here, we obtain any number of particular solutions. Since equation (4.28) is linear, the sum of all such par-

[1] It was derived by D'Alembert for a string, the equation of transverse vibration of which has the same form (4.8).

ticular solutions also satisfies it and we get its solution in the form of a series

$$u = \Sigma (A \cos \lambda a t + B \sin \lambda a t)(C \cos \lambda x + D \sin \lambda x); \quad (4.32)$$

A, B, C, D, λ may have their particular values in each term of this sum; the number of terms of the series is not limited.

The indefinite constants entering here should be sought from the boundary and initial conditions of the bar. This procedure will now be illustrated by a simple example. Let us consider a bar of length l with the lower end built in, performing longitudinal vibrations. The fastening condition at the lower end is: $u=0$ at any t and at $x=0$. To satisfy this condition it is obviously necessary to set in equation (4.32)

$$C = 0;$$

then

$$u = \Sigma (A \cos \lambda a t + B \sin \lambda a t) \sin \lambda x\,[1]. \quad (4.33)$$

Let us suppose that the upper end of the bar is free and the elongation is therefore equal to $e_{xx} = \frac{\partial u}{\partial x} = 0$ for $x = l$ at any instant of time t. Keeping this in mind, we obtain $\cos \lambda l = 0$ from equation (4.33). Hence

$$\lambda l = \frac{i\pi}{2} \quad \text{or} \quad \lambda = \frac{i\pi}{2l},$$

where i may be equal to any odd number: $i = 1, 3, 5, \ldots, \infty$ and, therefore,

$$u = \sum^{\infty} \left(A_i \cos \frac{i\pi}{2l} a t + B_i \sin \frac{i\pi}{2l} a t \right) \sin \frac{i\pi}{2l} x. \quad (4.34)$$

To determine the arbitrary constants A_i and B_i, it is necessary to specify the state of the bar at the initial moment, for example, at $t=0$, i.e., to indicate displacements u and velocities $\frac{\partial u}{\partial t}$ of all points of the bar at this instant; let

$$\left. \begin{aligned} u_{t=0} &= f(x), \\ \left(\frac{\partial u}{\partial t} \right)_{t=0} &= \varphi(x). \end{aligned} \right\} \quad (4.35)$$

[1] We can obviously omit the constant D by setting it equal to unity, without loss of generality of the solution, since by introducing it in parentheses we again obtain only the two constants AD and BD.

But

$$\frac{\partial u}{\partial t}=\frac{\pi a}{2l}\sum_{i=1,3,5,\ldots}^{\infty}\left(-A_i i\sin\frac{i\pi}{2l}\alpha t+B_i i\cos\frac{i\pi}{2l}\alpha t\right)\sin\frac{i\pi}{2l}x. \quad (4.36)$$

On the basis of equations (4.34) and (4.36), conditions (4.35) give

$$\left.\begin{aligned} f(x)&=\sum_{i=1}^{\infty}A_i\sin\frac{i\pi x}{2l},\\ \varphi(x)&=\frac{\pi a}{2l}\sum_{i=1}^{\infty}B_i i\sin\frac{i\pi x}{2l}. \end{aligned}\right\} \quad (4.37)$$

The problem has been reduced to the expansion of the functions $f(x)$ and $\varphi(x)$ into Fourier series appearing in the right-hand members of equations (4.37), i.e., to the determination of

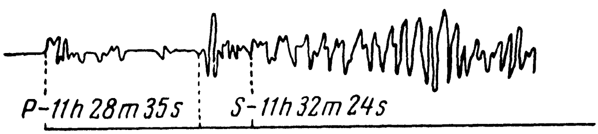

Fig. 36

coefficients A_i and B_i in these series. These coefficients, as is known, are expressed by Euler's formulas and have the following values in this case:

$$\left.\begin{aligned} A_i&=\frac{1}{l}\int_{-l}^{+l}f(x)\sin\frac{i\pi}{2l}x\,dx,\\ B_i&=\frac{2}{i\pi a}\int_{-l}^{+l}\varphi(x)\sin\frac{i\pi}{2l}x\,dx. \end{aligned}\right\} \quad (4.38)$$

The foregoing method is very effectual in the case of free vibrations of bars. Investigation of forced vibrations, as well as transverse vibrations of bars subjected to a periodically varying or moving load, makes the problem far more complicated.

The general theory of vibrations of an elastic body is widely used in seismology to study vibrations of the earthcrust; without going deep into this question, we shall note only that the observed constancy of the velocities of longitudinal and transverse vibrations makes it possible to determine from the record of

vibrations at the seismological station (seismogram) the distance between the epicentre and the station; transverse vibrations, as we have seen, travel more slowly than longitudinal vibrations and reach the station later; Fig. 36 represents a section of the seismogram of the earthquake in Asia Minor on February 9, 1909, recorded by the Pulkovo Observatory; here the points P and S show the moments of the arrival of the first longitudinal and the first transverse waves. Knowing the interval of time PS and the velocities of both kinds of waves, we can find the required distance. Some additional information on the propagation of elastic waves is given in Sections 56 and 57 of Chapter IX.

V

Solution of the Elasticity Problem in Terms of Stresses

29. THE SIMPLEST PROBLEMS

It was pointed out in Sec. 24 that if we wish directly to find stresses in an elastic body according to prescribed external forces

$$X_x,\ Y_y,\ Z_z,\ Y_z,\ Z_x,\ X_y, \tag{5.1}$$

the conditions of equilibrium (I)

$$\left.\begin{aligned}
&\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+\rho X=0;\\
&\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+\rho Y=0;\\
&\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+\rho Z=0
\end{aligned}\right\} \tag{I}$$

will be inadequate for this purpose, since the number of unknowns (5.1) exceeds the number of equations (I); it will, therefore, be necessary to resort to compatibility conditions (IV). With that end in view we shall further transform equations (IV), substituting stresses for strains [in accordance with equations (V) of Sec. 24].

Let us now consider the particular case when stresses (5.1) are expressed by functions of the first degree (linear functions) of the co-ordinates of a point or when stresses are constant. By proceeding from equations (V), we can easily show that the second derivatives of strains are always expressed by linear functions of the second derivatives of stresses (5.1), for instance,

$$\frac{\partial^2 e_{xx}}{\partial y^2}=\frac{1}{E}\left[\frac{\partial^2 X_x}{\partial y^2}-\sigma\left(\frac{\partial^2 Y_y}{\partial y^2}+\frac{\partial^2 Z_x}{\partial y^2}\right)\right],$$

$$\frac{\partial^2 e_{yz}}{\partial x\,\partial z}=\frac{1}{G}\frac{\partial^2 Y_z}{\partial x\,\partial z},\ \text{etc.}$$

Since the stresses are linear functions of x, y, z in our case, all the second derivatives of strains vanish; the compatibility conditions (IV) will thus all be satisfied. It remains only to satisfy equations (I) and the conditions on the surface of a body (II):

$$\left.\begin{aligned} X_v &= X_x \cos(vx) + X_y \cos(vy) + X_z \cos(vz); \\ Y_v &= Y_x \cos(vx) + Y_y \cos(vy) + Y_z \cos(vz); \\ Z_v &= Z_x \cos(vx) + Z_y \cos(vy) + Z_z \cos(vz). \end{aligned}\right\} \qquad \text{(II)}$$

Problems of this kind are called the simplest problems of the theory of elasticity. We shall now consider three such problems.

30. TORSION OF A CIRCULAR BAR

This problem deals with the state of stress and deformation of a bar having the form of a circular cylinder (Fig. 37) whose

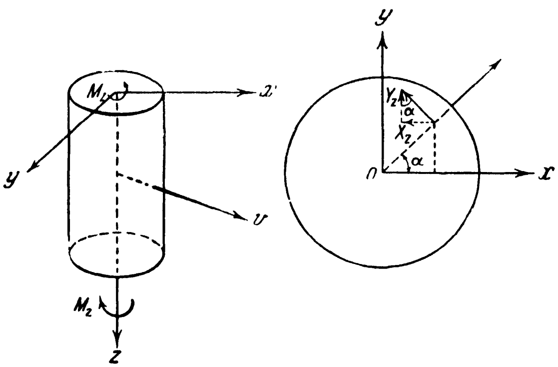

Fig. 37

bases are acted upon by external forces that produce only shearing stresses and reduce to two opposite couples; the moment of such a couple is called the *twisting moment (torque)*. According to the theory of torsion elaborated by C. A. Coulomb at the end of the 18th century, the deformation of a bar consists in rotations of the plane cross sections of the bar with respect to one another, involving no distortion (warping) of cross sections; at all points of the cross section there occur only shearing stresses T_z directed normally to radius-vectors of the points. These shearing stresses are assumed to be distributed similarly over all sections (including the ends of the cylinder); the system of forces acting at the

section reduces to the twisting moment

$$M_z = \int_F T_z r \, dF \tag{5.2}$$

where dF is an element of area isolated in the section; the integral is extended over the entire cross-sectional area. Let τ denote the angle of mutual rotation of two sections a distance equal to unity apart; from elementary geometrical considerations it is easy to find now the shear angle which occurs at a point of the section distant r from the axis of the section and which corresponds to the mutual angle of rotation τ:

$$1\gamma = r\tau.$$

Hence on the basis of Hooke's law in shear we obtain the magnitude of shearing stress T_z at the given point

$$T_z = G\gamma = Gr\tau. \tag{5.3}$$

Its projections on the x and y axes are (Fig. 37)

$$X_z = -T_z \sin\alpha = -T_z \frac{y}{r}; \quad Y_z = T_z \cos\alpha = T_z \frac{x}{r}$$

or, according to (5.3),

$$X_z = -G\tau y; \quad Y_z = G\tau x.$$

Assuming that the remaining stress components (5.1) vanish, we arrive at the following system of stresses:

$$\left.\begin{array}{lll} X_x = 0; & Y_y = 0; & Z_z = 0; \\ Y_z = G\tau x; & X_z = -G\tau y; & X_y = 0. \end{array}\right\} \tag{5.4}$$

Let us verify whether this system of stresses is possible from the viewpoint of the theory of elasticity. Since stresses (5.4) are functions of not higher than the first degree in the co-ordinates of a point, the problem under consideration is one of the simplest; it will, therefore, be necessary only to satisfy equations (I) and to check, as required by the surface conditions (II), whether stresses (5.4) correspond to the conditions of the torsion problem. Substituting the data from equations (5.4) in equations (I), we find

$$X = 0; \quad Y = 0; \quad Z = 0,$$

consequently, stresses (5.4) are possible provided body forces are absent (for instance, the own weight of a bar).

Let us now turn to the conditions on the surface of the bar. It can easily be seen (Fig. 37) that everywhere on the lateral

surface $\cos(vz) = 0$;

$$\cos(vx) = \cos\alpha = \frac{x}{r}; \quad \cos(vy) = \sin\alpha = \frac{y}{r}.$$

Substituting these values and those from equations (5.4) in equations (II), we find that

$$X_v = 0; \quad Y_v = 0; \quad Z_v = 0,$$

i.e., the lateral surface is free of load.

At the end cross section we have

$$\cos(vx) = \cos(vy) = 0,$$
$$\cos(vz) = 1.$$

Introducing these values and those from (5.4) in (II), we have

$$\left.\begin{aligned} &X_v = X_z = -G\tau y; \quad Y_v = Y_z = G\tau x; \\ &Z_v = 0. \end{aligned}\right\} \tag{5.5}$$

Consequently, only tangential tractions are applied here (as in any other cross section). Their resultant has the following projections on the axes Ox and Oy:

$$\int\int X_z\,dF = -\int\int G\tau y\,dF = -G\tau\int\int y\,dF = 0,$$
$$\int\int Y_z\,dF = G\tau\int\int x\,dF = 0,$$

since the origin of co-ordinates is placed at the centroid of the section. Consequently, the forces applied to the end section reduce to a couple; we thus really have the problem of the torsion of a bar.

The moment of the torsional couple about the axis Oz is

$$M_z = \int\int (X_z y - Y_z x)\,dF = -G\tau\int\int (x^2 + y^2)\,dF = -G\tau J_p, \tag{5.6}$$

where $J_p = \frac{\pi R^4}{2}$ is the polar moment of inertia of a circular section.

31. SAINT-VENANT'S PRINCIPLE

It is important to make the following remark. We have shown that the system of stresses (5.4) corresponds to the torsion of a circular bar; *we have no right, however, to draw the converse conclusion that the torsion of a circular bar always results in the system of stresses* (5.4).

Indeed, when formulating the problem of torsion, we indicate only that there is a couple applied to the end cross section; this

indication, however, is inadequate because the couple can be applied in a number of different ways, i.e., the stresses reducing to the couple can be distributed differently over the points of the section. The system of stresses (5.4) results when the torsional couple is applied in the specific manner indicated by formulas (5.5); if the torsional couple is applied in a different manner, another system of stresses will be set up in the bar.

These considerations also apply to many other problems; they show that we often formulate problems schematically, sometimes prescribing not the fòrces applied to a body but only their resultants (for instance, "compression of a bar by two forces" or "bending by a couple", etc.). Each of these formulations leads to a variety of quite different problems when a more exact method of solution is used.

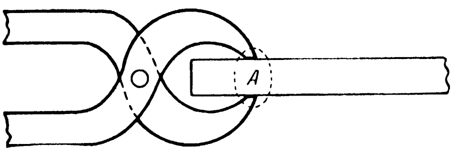

Fig. 38

Saint-Venant, however, propounded a very important general principle indicating that there is much in common in the infinite set of problems covered by such a general schematic formulation; this principle states that *at points in the solid which are at a sufficient distance from the surfaces of application of external loads, stresses depend very slightly on the particular manner in which these loads are applied*; for instance, in the case of the problem of torsion just solved Saint-Venant's principle states that at points in the bar sufficiently far remote from the end cross sections the stresses set up by twisting depend to a very little extent on the manner in which torques are applied and vary slightly when this manner is changed. But at the ends of the bar the mode of application of torques affects the nature and magnitude of stresses substantially.

Saint-Venant's principle results from the following postulate which is now sufficiently well proved both in the solution of many specific problems and in the general form.

If a balanced system of forces is applied to any part of a body A (Fig. 38), *it will induce stresses in this body which diminish very rapidly with the distance from the part* A.

Fig. 38 illustrates this postulate in the particular case of a bar which is compressed by pliers producing a balanced system of

forces. However great these forces may be (e.g., in cutting a wire with pliers), they will obviously cause no appreciable stresses outside the small region A outlined by the dotted line.

This postulate being accepted, we can pass on to Saint-Venant's principle formulated above by means of a simple static transformation. Our reasoning will again be illustrated by a concrete example (Fig. 39 a, b) of a rod bent in one case by a weight P suspended from the end of the rod from below, and in the other — by the same force but pressing from above.

It is evident that the stresses in the region A will be essentially different in these two cases, but it can easily be seen that the difference will be insignificant outside the region A.

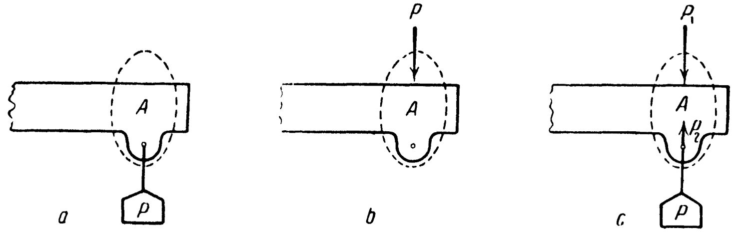

Fig. 39

Indeed, let us consider an auxiliary third case (Fig. 39c), adding two equal and opposite forces P_1 and P_2 to case a, the three forces being the same in modulus: $P_1 = P_2 = P$.

We observe that case c differs

from case a by the balanced system of forces P_1, P_2,

from case b by the balanced system of forces P, P_2.

These balanced systems, however, will cause no appreciable stresses outside the small region A, in accordance with the above-stated principle; consequently, case c will give a very slight difference in stresses outside this region in comparison with cases a and b; thus, cases a and b will also be almost identical with regard to stresses outside the region A, which was to be proved.

It should be pointed out that the objective of the foregoing considerations is to give a general idea of Saint-Venant's principle in view of its exceptional importance for the statement of problems of the theory of elasticity and for their solution. However, the above schematic formulation of the postulate concerning

a balanced system of forces applied to a part of a body may lead to erroneous application of Saint-Venant's principle, unless we introduce an essential amendment; taking this into consideration, we shall give a more complete formulation of the conditions in which Saint-Venant's principle is valid.

Let us take on the surface of a body a small region *whose dimensions are not great compared with the overall dimensions of the body*; let a balanced load (i.e., a load whose resultant and moment are equal to zero) be distributed over this region; Saint-Venant's principle states that under the action of such a load stresses will be appreciable only in the part of the body in the

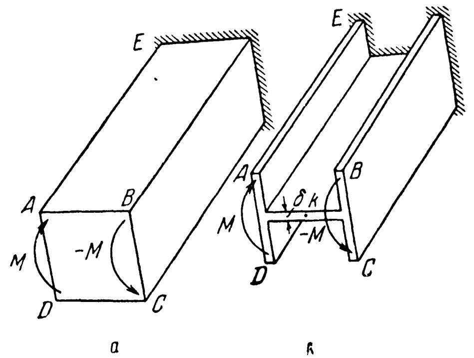

Fig. 40

immediate vicinity of the loaded region; they diminish so quickly that at points whose distance from the loaded region exceeds the greatest dimension of this region they can practically be neglected.

It is essential in this formulation that the loaded region be small in comparison with the *overall* dimensions of a body; in the limiting case it may be *of the same order as the smallest dimension of a body*. If this requirement is not fulfilled, the conditions in which Saint-Venant's principle is valid are not satisfied and it cannot be applied. This consideration must be kept in mind in the case of thin shells and the so-called thin-walled bars, such as bars of I-section (Fig. 40*b*). Here the dimensions characterising the shape of a body may differ widely; for instance, the overall dimensions *AB* and *BC* of the cross section may be small as compared with the length of the bar *AE*, but considerably greater than the web thickness δ. Take two bars (Fig. 40) built into a wall in the same manner, one of them with a rectangular section and the other with an I-section, the dimensions *AB*, *BC* and

AE being equal in both cases. Let identical balanced systems of forces consisting of two couples with equal but opposite moments $+M$ and $-M$ be applied to the free end of each bar. If the cross-sectional dimensions AB and BC are small compared to the length of the bar AE, then, according to Saint-Venant's principle, we can state that in the case represented in Fig. 40*a* the stresses developed in the bar will be practically negligible except for the small portion adjacent to the free end; the length of this portion will be of the same order as the cross-sectional dimensions AB and BC.

In the case represented in Fig. 40*b* this cannot be affirmed, since the web thickness δ of the cross section is small in comparison with the dimensions of the loaded region $ABCD$. This is evident from the following considerations: let us imagine that the web thickness δ is made to approach zero; in the limit the web will disappear and the bar will be converted into two separate strips (flanges of the I-bar), each loaded by an unbalanced system of forces, i.e., by a couple; each strip will undergo pure bending to be considered in the next section; the stresses will be the same in all cross sections of each strip and will extend up to its fixed end, no matter how great its length may be. The web thickness δ is not in reality zero and there will be no such sharp effect; however, the smaller the δ, the greater the region of extension of stresses along the length of the bar. In spite of this, it cannot be said that in the case represented in Fig. 40*b* Saint-Venant's principle is not true: there are simply no conditions here for the principle to be applicable. Such conditions will be fulfilled, if we draw the couples $+M$ and $-M$ nearer to each other and apply them to the web of the I-bar at some point K so that the distance between the planes of the couples does not exceed δ, the smallest dimension of the bar. Then Saint-Venant's principle will be valid and it may be stated that the stresses will be negligible everywhere except for a small region in the neighbourhood of the point K; the dimensions of this region are of the same order as the web thickness δ.

32. THE PROBLEM OF TORSION OF A CIRCULAR BAR (CONTINUED)

Let us finish the problem of the torsion of a circular bar and find the displacements of its points.

Introducing stresses (5.4) in Hooke's law (V) (Sec. 24), we obtain strains; we then replace the strains by their expressions in terms of displacements in accordance with equations (III)

(Sec. 24) and obtain the following system of equations:

$$\frac{\partial u}{\partial x}=0, \quad \frac{\partial v}{\partial y}=0, \quad \frac{\partial w}{\partial z}=0, \tag{5.7}$$

$$\left.\begin{aligned} &\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0,\\ &\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\tau x,\\ &\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=-\tau y. \end{aligned}\right\} \tag{5.8}$$

Integrating equations (5.7), we have

$$\left.\begin{aligned} &u=f(y, z), \quad v=\varphi(x, z),\\ &w=\psi(x, y), \end{aligned}\right\} \tag{5.9}$$

where f, φ, ψ are arbitrary functions; their form will be determined from equations (5.8) which, on the basis of (5.9), become

$$\left.\begin{array}{|c|c|c|l} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & \dfrac{\partial \varphi}{\partial x}+\dfrac{\partial f}{\partial y}=0,\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} & \dfrac{\partial \psi}{\partial y}+\dfrac{\partial \varphi}{\partial z}=\tau x,\\ \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial f}{\partial z}+\dfrac{\partial \psi}{\partial x}=-\tau y. \end{array}\right\} \tag{5.10}$$

Let us attempt to eliminate functions f, φ, ψ in succession from these equations by means of differentiation. This process can be carried out systematically; the succession of the required computations will be given in detail as it will be employed in the same form in other problems treated below in Sections 33 and 34. Let us differentiate equations (5.10) successively, as indicated in the first, second and third columns to the left of these equations; we obtain the following nine equations:

$$\left.\begin{array}{l} \dfrac{\partial^2 \varphi}{\partial x^2}=0; \quad \dfrac{\partial^2 \psi}{\partial y^2}=0; \quad \dfrac{\partial^2 f}{\partial z^2}=0;\\ \dfrac{\partial^2 f}{\partial y^2}=0; \quad \dfrac{\partial^2 \varphi}{\partial z^2}=0; \quad \dfrac{\partial^2 \psi}{\partial x^2}=0;\\ \begin{array}{|c|c|c|l} + & + & - & \dfrac{\partial^2 \varphi}{\partial x\, \partial z}+\dfrac{\partial^2 f}{\partial y\, \partial z}=0;\\ + & - & + & \dfrac{\partial^2 \psi}{\partial x\, \partial y}+\dfrac{\partial^2 \varphi}{\partial x\, \partial z}=\tau;\\ - & + & + & \dfrac{\partial^2 f}{\partial y\, \partial z}+\dfrac{\partial^2 \psi}{\partial x\, \partial y}=-\tau. \end{array} \end{array}\right\} \tag{5.11}$$

We add up the last three of these equations by members thrice, changing the signs each time, as indicated in three columns to the left; we obtain the following three equalities:

$$2\frac{\partial^2\varphi}{\partial x\,\partial z}=2\tau;\quad 2\frac{\partial^2 f}{\partial y\,\partial z}=-2\tau;\quad 2\frac{\partial^2\psi}{\partial x\,\partial y}=0. \tag{5.12}$$

Thus, equations (5.11) and (5.12) give the following table of the second derivatives of functions (5.9):

$$\left.\begin{aligned}
&\frac{\partial^2 f}{\partial y^2}=0; && \frac{\partial^2 f}{\partial y\,\partial z}=-\tau; && \frac{\partial^2 f}{\partial z^2}=0;\\
&\frac{\partial^2\varphi}{\partial x^2}=0; && \frac{\partial^2\varphi}{\partial x\,\partial z}=\tau; && \frac{\partial^2\varphi}{\partial z^2}=0;\\
&\frac{\partial^2\psi}{\partial x^2}=0; && \frac{\partial^2\psi}{\partial x\,\partial y}=0; && \frac{\partial^2\psi}{\partial y^2}=0.
\end{aligned}\right\} \tag{5.13}$$

The object of the preceding computations was to obtain precisely this, simpler system of equations in place of (5.10); based on it, the general form of functions f, φ and ψ can easily be established now; in fact, the last three equations show that the function ψ will be of the first degree in x and y; the first and second lines show that f and φ must be of the second degree (in y, z and in x, z, respectively) but must not contain the squares of the variables; therefore, on the basis of equations (5.13) and (5.9) we can easily write

$$\left.\begin{aligned}
f&=-\tau yz+ay+bz+c,\\
\varphi&=\tau xz+dx+ez+f,\\
\psi&=gx+hy+k.
\end{aligned}\right\} \tag{5.14}$$

Functions (5.14) are the general solution of equations (5.13); these latter have been obtained as a consequence of equations (5.10) by means of differentiation. Their order, therefore, is higher than the order of the latter; thus, it is not every solution of equations (5.13) which will satisfy equations (5.10). For instance, by differentiating the equation

$$\frac{dy}{dx}=0,$$

we obtain

$$\frac{d^2y}{dx^2}=0.$$

The general solution of the latter equation

$$y=C_1x+C_2$$

will satisfy the first equation only if $C_1=0$. It is, therefore, necessary to ascertain which of the solutions (5.14) satisfy equations (5.10).

Substituting functions (5.14) in equations (5.10), we establish the following relationships:

$$\tau z+d-\tau z+a=0,$$
$$h+\tau x+e=\tau x,$$
$$-\tau y+b+g=-\tau y,$$

or

$$d=-a; \quad h=-e; \quad g=-b,$$

and finally we have

$$\left.\begin{aligned} u&=f=-\tau yz+ay+bz+c,\\ v&=\varphi=\tau xz-ax+ez+f,\\ w&=\psi=-bx-ey+k. \end{aligned}\right\} \tag{5.15}$$

The obtained expressions of displacements (5.15) involve six arbitrary constants whose presence here can easily be explained. Indeed, in Sec. 10 we agreed to fix the elastic body under investigation in order to eliminate its motion dealt with in mechanics of absolutely rigid bodies and defined above, at the end of Sec. 10, as a rigid body displacement; in the present problem, however, we have not yet realised such fixing; the linear trinomials in formulas (5.15) indicate that it is still possible for the rod to rotate as a whole about the co-ordinate axes through arbitrary small angles $-e$, $-b$, $-a$ and to have translatory displacements c, f, k along these axes [cf. formulas (2.11)].

Let the fixing conditions be realised at the upper end of the bar: fix the centroid of the section $z=0$, requiring that

$$\left.\begin{aligned} &\text{for}\quad x=y=z=0\\ &u=0;\ v=0;\ \ w=0. \end{aligned}\right\} \tag{5.16}$$

This condition, however, is inadequate as, in spite of it, the bar can rotate about the origin (Fig. 41); in order to eliminate the possibility of rotation let us require that two of three elements dx, dy and dz near the origin of co-ordinates O remain immovable. In Sec. 10 [see formulas (2.3) and (2.4) and matrix (2.8)] we already obtained the expressions of the angles of rotation of elementary segments; by using them, we can write down the required conditions.

(a) The segment dz is immovable

$$\text{for } x = y = z = 0,$$

$$\frac{\partial u}{\partial z} = 0; \quad \frac{\partial v}{\partial z} = 0. \tag{5.17}$$

(b) The segment dy is immovable in the plane Oxy

$$\text{for } x = y = z = 0, \quad \frac{\partial u}{\partial y} = 0. \tag{5.18}$$

It is easy to conceive that conditions (5.16), (5.17) and (5.18) are sufficient to prevent a rigid body displacement of the bar.

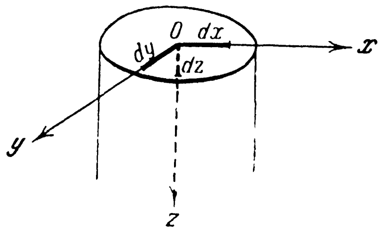

Fig. 41

Applying these conditions to equations (5.15), we obtain

$$a = b = c = e = f = k = 0\,[1],$$

and therefore

$$\left.\begin{aligned} u &= -\tau yz, \\ v &= \tau xz, \\ w &= 0. \end{aligned}\right\} \tag{5.19}$$

The last equation supports the hypothesis of plane cross sections of a circular bar remaining plane in torsion.

33. PURE BENDING OF A PRISMATICAL BAR

We consider a bar of prismatical cross section (Fig. 42) with equal but opposite couples applied to its ends. Let the z axis be directed along the axis of the bar; the xz plane coincides with the plane of action of the applied couples. This case is commonly referred to as pure bending; its elementary theory was elaborated by J. Bernoulli and L. Euler in the 18th century; it is based on

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[1] Thus all components of a rigid body displacement in formulas (2.11) vanish.

the assumption that the axis of the bar OB will bend along a curve lying in the xz plane and that plane cross sections of the bar will remain plane and normal to the deflected axis. From simple geometrical considerations (as set forth in the courses on strength of materials) it may be concluded that longitudinal "fibres" of the bar parallel to its axis receive unit elongations proportional to their distances from the yz plane containing the axis of the bar, which is called the *neutral* plane

$$e_{zz} = -\frac{x}{\rho},$$

where ρ is the radius of curvature of the deflected axis of the bar. It is assumed that all components of the stress tensor are zero except for the component Z_z causing simple tension or compression of the fibres of the bar; then, in accordance with Hooke's law we find that

$$Z_z = Ee_{zz} = -\frac{Ex}{\rho},$$

where E is Young's modulus. On the basis of the above we arrive at the following system of stresses:

$$X_x = Y_y = X_y = Y_z = Z_x = 0; \quad Z_z = -\frac{Ex}{\rho}. \qquad (5.20)$$

Let us see whether the system of stresses (5.20) is possible from the viewpoint of the theory of elasticity and whether these stresses correspond to pure bending.

By substituting functions (5.20) in the equations of equilibrium (I), we see that they are satisfied in the absence of body forces. The equations of continuity (IV) (Sec. 24) will be satisfied. As regards the conditions on the lateral surface of the bar we have everywhere

$$\cos(vz) = 0.$$

Substituting this and (5.20) in equations (II), we find that the lateral surface is free of stresses.

The forces on the end section B reduce to a couple[1] with the moment

$$M_y = \int\int Z_z x\,dF = -\frac{E}{\rho}\int\int x^2\,dF = -\frac{EJ}{\rho}.$$

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[1] The reader is recommended to confirm for himself that the resultant of forces $Z_z dF$ is equal to zero.

Consequently, stresses (5.20) are possible and correspond to pure bending[1]. We proceed to the determination of displacements. Substituting stresses (5.20) in Hooke's law (V) and the strains thus obtained in equations (III), we arrive at the system of equations:

$$\frac{\partial u}{\partial x}=\frac{\sigma x}{\rho};\quad \frac{\partial v}{\partial y}=\frac{\sigma x}{\rho};\quad \frac{\partial w}{\partial z}=-\frac{x}{\rho};\tag{5.21}$$

$$\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=0;\quad \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0;\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0.\tag{5.22}$$

They are integrated in the same manner as in the preceding section. From equations (5.21) we find

$$\left.\begin{aligned} u&=\frac{\sigma x^2}{2\rho}+f(y,\,z),\\ v&=\frac{\sigma xy}{\rho}+\varphi(x,\,z),\\ w&=-\frac{xz}{\rho}+\psi(x,\,y).\end{aligned}\right\}\tag{5.23}$$

The form of functions f, φ and ψ is determined by the use of equations (5.22):

$$\left.\begin{aligned} &\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial z}=0,\\ &\frac{\partial f}{\partial z}+\frac{\partial\psi}{\partial x}=\frac{z}{\rho},\\ &\frac{\partial\varphi}{\partial x}+\frac{\partial f}{\partial y}=-\frac{\sigma y}{\rho}.\end{aligned}\right\}\tag{5.24}$$

These equations have the same form as equations (5.10) in the problem of torsion; consequently, to determine functions f, φ, ψ we can apply the procedure given above; in this way we find the second derivatives:

$$\begin{aligned} &\frac{\partial^2 f}{\partial y^2}=-\frac{\sigma}{\rho}; &&\frac{\partial^2 f}{\partial z^2}=\frac{1}{\rho}; &&\frac{\partial^2 f}{\partial y\,\partial z}=0;\\ &\frac{\partial^2\varphi}{\partial x^2}=0; &&\frac{\partial^2\varphi}{\partial z^2}=0; &&\frac{\partial^2\varphi}{\partial x\,\partial z}=0;\\ &\frac{\partial^2\psi}{\partial x^2}=0; &&\frac{\partial^2\psi}{\partial y^2}=0; &&\frac{\partial^2\psi}{\partial x\,\partial y}=0.\end{aligned}$$

With these expressions we readily determine the form of functions f, φ, ψ and, substituting the result in equations (5.23), we

[1] It is to be noted that the foregoing theory is valid only when the plane xz of action of the couples is a *principal plane* of the bar, i.e., it contains one of the principal axes of inertia of each section of the bar.

obtain these in the following form:

$$\left.\begin{aligned} u &= \frac{z^2}{2\rho} + \frac{\sigma(x^2 - y^2)}{2\rho} + ay + bz + c, \\ v &= \frac{\sigma xy}{\rho} - ax + ez + f', \\ w &= -\frac{xz}{\rho} - bx - ey + k. \end{aligned}\right\} \tag{5.25}$$

The linear trinomials included here represent the rigid body displacement of the bar, just as in the problem of torsion. To determine them let us fix the bar, for example, at its left-hand end

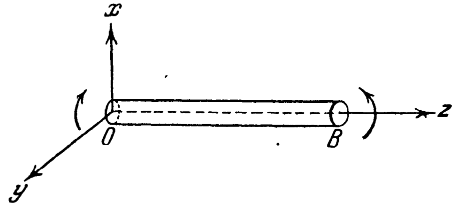

Fig. 42

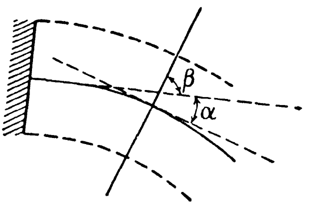

Fig. 43

(Fig. 42). In the first instance we fix the centroid of the section, requiring that

$$\text{for } x = y = z = 0$$

$$u = 0, \quad v = 0, \quad w = 0;$$

on this basis from equations (5.25) we find

$$c = f' = k = 0.$$

We fix further the element dz of the bar axis eliminating its rotations in the planes Oxy and Oyz, i.e., requiring that

$$\text{for } x = y = z = 0 \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0.$$

Finally, it remains to eliminate the rotation of the bar about its axis; for that we fix, for instance, the element dx of the axis Ox to prevent it from rotating in the plane Oxy, i.e., we require that

$$\text{for } x = y = z = 0 \quad \frac{\partial v}{\partial x} = 0.$$

From all these conditions we find that $a=b=e=0$ and we obtain the final expressions for displacements:

$$\left.\begin{aligned} u &= \frac{z^2+\sigma(x^2-y^2)}{2\rho}, \\ v &= \frac{\sigma xy}{\rho}, \\ w &= -\frac{xz}{\rho}. \end{aligned}\right\} \tag{5.26}$$

Each point (x_1, y_1, z_1) of the bar moves to a new position after deformation:

$$\begin{aligned} x &= x_1+u = x_1+\frac{z_1^2+\sigma\left(x_1^2-y_1^2\right)}{2\rho}, \\ y &= y_1+v = y_1+\frac{\sigma x_1 y_1}{\rho}, \\ z &= z_1+w = z_1-\frac{x_1 z_1}{\rho}. \end{aligned}$$

For points of the axis of the bar $(x=y=0)$ we obtain

$$\left.\begin{aligned} v &= w = 0, \\ u &= \frac{z^2}{2\rho}; \end{aligned}\right\} \tag{5.27}$$

this is the equation of the deflected axis of the bar.

Let us take some plane section of the bar $z=z_0$. The new coordinate z of any of its points after deformation will be

$$z = z_0 + w_0;$$

but in accordance with equations (5.26) we have

$$w_0 = -\frac{xz_0}{\rho}.$$

Consequently,

$$z = z_0\left(1-\frac{x}{\rho}\right). \tag{5.28}$$

This is the equation of a plane parallel to the axis Oy (Fig. 43); it is satisfied by all points of the section $z=z_0$ after deformation. Consequently, a plane cross section remains plane after deformation.

We rewrite equation (5.28) as

$$x = -\frac{\rho}{z_0}z+\rho. \tag{5.29}$$

The angular coefficient will be $\tan\beta = -\frac{\rho}{z_0}$. The slope of the tangent to axis (5.27) at the point $z=z_0$ is

$$\tan\alpha = \left(\frac{du}{dz}\right)_{z=z_0} = \frac{z_0}{\rho}. \tag{5.30}$$

Thus we have (Fig. 43)

$$\tan\alpha \cdot \tan\beta = -1.$$

Consequently, *the plane section $z=z_0$ remains normal to the axis after bending and the hypothesis of plane cross sections is fully justified in the case of pure bending.*

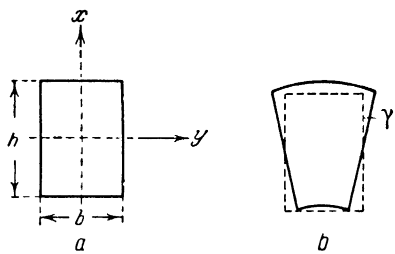

Fig. 44

Consider another particular case of a bar of rectangular cross section (Fig. 44 *a*) and investigate the distortion of the contour of its cross section. This question is of interest for what follows.

Its lateral side before bending is defined by the equations

$$z=z_0; \quad y = \pm\frac{b}{2};$$

after deformation these equations become

$$y = \pm\frac{b}{2} + v = \pm\frac{b}{2} \pm \frac{\sigma x b}{2\rho} = \pm\frac{b}{2}\left(1+\frac{\sigma x}{\rho}\right),$$

$$z = z_0 + w = z_0 - \frac{x z_0}{\rho} = z_0\left(1-\frac{x}{\rho}\right).$$

There result two equations of the first degree in x, y, z; consequently, the lateral sides remain straight after bending. From the equation

$$y = \pm\frac{\sigma b}{2\rho}x \pm \frac{b}{2}$$

we see that (Fig. 44 *b*)

$$\tan\gamma = \frac{\sigma b}{2\rho}, \tag{5.31}$$

where γ is the angle between the projections of the original and deformed positions of the lateral side on the plane Oxy. Now consider the upper and lower sides given by the equations

$$z = z_0; \quad x = \pm \frac{h}{2}.$$

After bending, their equations will be

$$x = \pm \frac{h}{2} + u = \pm \frac{h}{2} + \frac{z_0^2 + \sigma\left(\frac{h^2}{4} - y^2\right)}{2\rho}; \quad z = z_0\left(1 - \frac{x}{\rho}\right).$$

The first of these equations is of the second degree, while the second equation is of the first degree in x, y, z. Consequently, the upper and lower sides become plane curves. The first equation gives the projection of these curves on the plane Oxy:

$$x = \pm \frac{h}{2} + \frac{z_0^2}{2\rho} + \frac{\sigma h^2}{8\rho} - \frac{\sigma}{2\rho} y^2. \tag{5.32}$$

This is the equation of two parabolas (the upper and lower sides). The radius of curvature of the deflected axis (5.27) and the radius of curvature of parabolas (5.32) are approximately equal to

ρ and $\frac{\rho}{\sigma}$, respectively.

Equations (5.27) and (5.32) are approximate, since we have disregarded finite small quantities of higher order in the expressions of strains [cf. Sec. 10, formulas (2.3) and (2.4)]. A more exact solution shows curves (5.27) and (5.32) to be circles with radii

ρ and $\frac{\rho}{\sigma}$.

The distorted form of the section contour is represented in Fig. 44 *b*. The upper broadened part corresponds to compressed longitudinal fibres, the lower part—to stretched ones.

34. PRISM STRETCHED BY ITS OWN WEIGHT

Let us have (Fig. 45 *a*) a prism or a cylinder of arbitrary cross section suspended from the upper end and stretched by its own weight. It will immediately be seen that the following system of stresses corresponds to the given problem:

$$X_x = Y_y = 0; \quad Z_z = \rho g z; \quad Y_z = Z_x = X_y = 0. \tag{5.33}$$

Substituting these expressions in equations (I), we find that $X=Y=0$, $Z=-g$, i.e., we actually have the problem involving the action of the own weight. Furthermore, at $z=0$ we get

$$Z_z=0,$$

i.e., the lower end is free of load; introducing values (5.33) in equations (II), we see that the lateral surface of the prism is also free of load. Displacements are sought in exactly the same way as in the two previous problems.

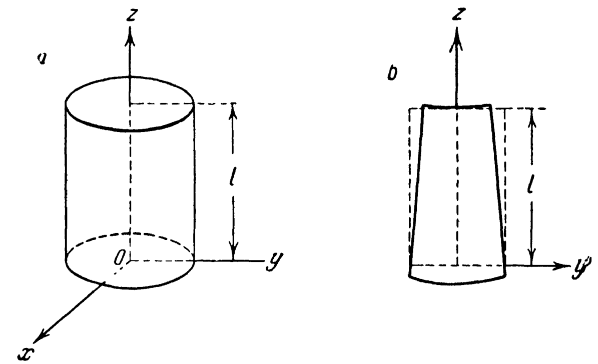

Fig. 45

On the basis of (III) and (V) we obtain the system of differential equations

$$\left.\begin{aligned}&\frac{\partial u}{\partial x}=-\frac{\sigma\rho g}{E}z, \quad \frac{\partial v}{\partial y}=-\frac{\sigma\rho g}{E}z,\\ &\frac{\partial w}{\partial z}=\frac{\rho g}{E}z;\end{aligned}\right\} \tag{5.34}$$

$$\left.\begin{aligned}&\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0, \quad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0,\\ &\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=0.\end{aligned}\right\} \tag{5.35}$$

By integrating equations (5.34), we get

$$\left.\begin{aligned}u&=-\frac{\sigma\rho g}{E}zx+f(y,\ z),\\ v&=-\frac{\sigma\rho g}{E}zy+\varphi(x,\ z),\\ w&=\frac{\rho g}{E}\frac{z^2}{2}+\psi(x,\ y).\end{aligned}\right\} \tag{5.36}$$

Introducing this in (5.35), we obtain the equations for the determination of f, φ and ψ:

$$\left.\begin{aligned} \frac{\partial f}{\partial y}+\frac{\partial \varphi}{\partial x}&=0,\\ -\frac{\sigma\rho g}{E}y+\frac{\partial \varphi}{\partial z}+\frac{\partial \psi}{\partial y}&=0,\\ \frac{\partial \psi}{\partial x}-\frac{\sigma\rho g}{E}x+\frac{\partial f}{\partial z}&=0. \end{aligned}\right\} \tag{5.37}$$

The method of determining them remains similar to that used in Sections 32 and 33 and, therefore, without repeating it, we present the final result:

$$\left.\begin{aligned} u&=-\frac{\sigma\rho g}{E}zx+ay+bz+c,\\ v&=-\frac{\sigma\rho g}{E}zy-ax+ez+m,\\ w&=\frac{\rho g}{2E}[z^2+\sigma(x^2+y^2)]-bx-ey+k. \end{aligned}\right\} \tag{5.38}$$

The fixing conditions of the prism are realised in the same way as in the two preceding problems, but according to the conditions of this problem we fix the centroid of the upper section rather than the co-ordinate origin; we require, therefore, that

$$\text{for } x=0,\ y=0,\ z=l$$

there must be

$$u=0;\qquad v=0;\qquad w=0;$$
$$\frac{\partial u}{\partial z}=0;\qquad \frac{\partial v}{\partial z}=0;\qquad \frac{\partial v}{\partial x}=0\,[1].$$

Hence we shall easily find that $a=b=c=e=m=0$, $k=-\frac{\rho g}{2E}l^2$ and displacements (5.38) will finally take the form

$$\left.\begin{aligned} u&=-\frac{\sigma\rho g}{E}xz;\quad v=-\frac{\sigma\rho g}{E}yz;\\ w&=\frac{\rho g}{2E}[z^2-l^2+\sigma(x^2+y^2)]. \end{aligned}\right\} \tag{5.39}$$

The points on the axis of the bar (its equations: $x=0$, $y=0$) displace in a vertical direction ($u=0$; $v=0$); all other points have horizontal displacements too.

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[1] One may require instead that $\frac{\partial u}{\partial y}=0$.

If we take a point x_0, y_0, z_0 in the unstrained state, its co-ordinates, as a result of deformation, will be

$$\left.\begin{aligned} x &= x_0\left(1-\frac{\sigma\rho g}{E}z_0\right), \\ y &= y_0\left(1-\frac{\sigma\rho g}{E}z_0\right), \\ z &= z_0+\frac{\rho g}{2E}\left[z_0^2-l^2+\sigma\left(x_0^2+y_0^2\right)\right]. \end{aligned}\right\} \tag{5.40}$$

Let us use them to investigate the character of deformation of the prism.

1. Consider its cross section

$$z = z_0 = \text{const.}$$

By squaring the first two of equations (5.40) and adding them up, we obtain

$$x^2+y^2=\left(1-\frac{\sigma\rho g}{E}z_0\right)^2\left(x_0^2+y_0^2\right) \tag{5.41}$$

or

$$r^2=\left(1-\frac{\sigma\rho g}{E}z_0\right)^2 r_0^2,$$

where $r^2=x^2+y^2$, $r_0^2=x_0^2+y_0^2$; thus each circle of radius r_0 at the cross section with centre on the axis of the prism is transformed into a circle of radius r with $r<r_0$. The surface into which the section $z=z_0$ is transformed will be found by eliminating the initial co-ordinates x_0 and y_0 from (5.40); taking into account (5.41), we obtain from the third equation of (5.40)

$$z=z_0+\frac{\rho g}{2E}\left(z_0^2-l^2+\frac{\sigma}{\left(1-\frac{\sigma\rho g}{E}z_0\right)^2}r^2\right). \tag{5.42}$$

This equation gives a paraboloid of revolution in cylindrical co-ordinates r, z, φ since the polar angle φ is not involved here.

2. Imagine a circular cylindrical surface of radius r_0 isolated in the prism

$$x_0^2+y_0^2=r_0^2. \tag{5.43}$$

As a result of deformation it will be transformed into a surface of revolution; this could be predicted on the basis of the above considerations concerning the deformation of circles described by equation (5.41). To obtain the equation of this surface it is necessary to eliminate the initial co-ordinates x_0, y_0, z_0 from four equations (5.40) and (5.43); the first two of equations (5.40),

however, can be replaced by equation (5.41):

$$r^2=\left(1-\frac{\sigma\rho g}{E}z_0\right)^2 r_0^2,$$

$$z=z_0+\frac{\rho g}{2E}\left[z_0^2-l^2+\sigma(x_0^2+y_0^2)\right],$$

$$x_0^2+y_0^2=r_0^2.$$

Hence we eliminate $x_0^2+y_0^2$ at once; there remain two equations

$$\left.\begin{aligned} r&=\left(1-\frac{\sigma\rho g}{E}z_0\right)r_0,\\ z&=z_0+\frac{\rho g}{2E}(z_0^2-l^2+\sigma r_0^2).\end{aligned}\right\}\qquad(5.44)$$

To obtain the equation of the required surface it remains to eliminate z_0 from them (or to treat them as parametric equations of a surface with the parameter z_0; these equations give the generator of our surface in co-ordinates r, z). The first of equations (5.44) gives

$$z_0=\frac{E}{\sigma\rho g r_0}(r_0-r).$$

Substituting this in the second equation, we get

$$z=\frac{E}{\sigma\rho g r_0}(r_0-r)+\frac{E}{2\sigma^2\rho g r_0^2}(r_0-r)^2+\frac{\sigma\rho g}{2E}r_0^2.\qquad(5.45)$$

This is a surface obtained by rotating a parabola about the z axis.

35. UNIQUENESS OF SOLUTION OF ELASTICITY EQUATIONS

In the solution of the preceding problems we used the inverse method prescribing stresses and determining what forces acting on the surface cause the selected system of stresses; then each time the question may arise whether or not it is possible to obtain the same forces on the surface for some other system of stresses. If so, the solution of equations of the theory of elasticity will be multiple-valued: several systems of stresses will correspond to given forces on the surface and it will be necessary to clarify which of the systems actually take place. In this case, using either the inverse or semi-inverse method of solution, we shall not be certain that we have chosen the system of stresses which corresponds to reality. Accordingly, the question of uniqueness of solution of elasticity equations acquires great importance.

It is to be noted that if initial stresses are possible in a body, then under the action of external forces on this body the total stresses developed in it may differ very widely; take, for instance, the iron sphere with the initial stresses spoken of in Sec. 21.

If we subject it to the action of forces, the final overall stresses in it will depend on the dimension of the sectorial cut-out which has been previously made. In cases like these, the equations of the theory of elasticity must have a multiple-valued solution.

Let us omit the case of initial stresses, i.e., accept *the hypothesis of the natural state* of a body, and also accept the principle of superposition. It is easy to prove, then, that the solution will be single-valued and, therefore, unique.

Indeed, let us assume the converse, namely, that under the action of given surface tractions

$$X_v, \ Y_v, \ Z_v \tag{5.46}$$

and body forces

$$X, \ Y, \ Z \tag{5.47}$$

two different systems of stresses are possible:

$$X'_x, \ Y'_y, \ Z'_z, \ Y'_z, \ Z'_x, \ X'_y \tag{5.48$'$}$$

and

$$X''_x, \ Y''_y, \ Z''_z, \ Y''_z, \ Z''_x, \ X''_y. \tag{5.48$''$}$$

Both of these systems must satisfy basic equations (I) and (II):

$$\left.\begin{aligned} &\frac{\partial X'_x}{\partial x}+\frac{\partial X'_y}{\partial y}+\frac{\partial X'_z}{\partial z}+\rho X=0,\\ &\frac{\partial Y'_x}{\partial x}+\frac{\partial Y'_y}{\partial y}+\frac{\partial Y'_z}{\partial z}+\rho Y=0,\\ &\frac{\partial Z'_x}{\partial x}+\frac{\partial Z'_y}{\partial y}+\frac{\partial Z'_z}{\partial z}+\rho Z=0; \end{aligned}\right\} \tag{5.49$'$}$$

$$\left.\begin{aligned} X_v&=X'_x\cos(vx)+X'_y\cos(vy)+X'_z\cos(vz),\\ Y_v&=Y'_x\cos(vx)+Y'_y\cos(vy)+Y'_z\cos(vz),\\ Z_v&=Z'_x\cos(vx)+Z'_y\cos(vy)+Z'_z\cos(vz); \end{aligned}\right\} \tag{5.50$'$}$$

$$\left.\begin{aligned} &\frac{\partial X''_x}{\partial x}+\frac{\partial X''_y}{\partial y}+\frac{\partial X''_z}{\partial z}+\rho X=0,\\ &\frac{\partial Y''_x}{\partial x}+\frac{\partial Y''_y}{\partial y}+\frac{\partial Y''_z}{\partial z}+\rho Y=0,\\ &\frac{\partial Z''_x}{\partial x}+\frac{\partial Z''_y}{\partial y}+\frac{\partial Z''_z}{\partial z}+\rho Z=0; \end{aligned}\right\}. \tag{5.49$''$}$$

$$\left.\begin{aligned} X_v&=X''_x\cos(vx)+X''_y\cos(vy)+X''_z\cos(vz),\\ Y_v&=Y''_x\cos(vx)+Y''_y\cos(vy)+Y''_z\cos(vz),\\ Z_v&=Z''_x\cos(vx)+Z''_y\cos(vy)+Z_z\cos(vz). \end{aligned}\right\} \tag{5.50$''$}$$

By subtracting the respective equations of systems (5.49) and (5.50) by members, we obtain the following new system of equations:

$$\left.\begin{aligned}
&\frac{\partial(X'_x - X''_x)}{\partial x} + \frac{\partial(X'_y - X''_y)}{\partial y} + \frac{\partial(X'_z - X''_z)}{\partial z} = 0,\\
&\frac{\partial(Y'_x - Y''_x)}{\partial x} + \frac{\partial(Y'_y - Y''_y)}{\partial y} + \frac{\partial(Y'_z - Y''_z)}{\partial z} = 0,\\
&\frac{\partial(Z'_x - Z''_x)}{\partial x} + \frac{\partial(Z'_y - Z''_y)}{\partial y} + \frac{\partial(Z'_z - Z''_z)}{\partial z} = 0;\\
&0 = (X'_x - X''_x)\cos(vx) + (X'_y - X''_y)\cos(vy) + \\
&\qquad\qquad + (X'_z - X''_z)\cos(vz),\\
&0 = (Y'_x - Y''_x)\cos(vx) + (Y'_y - Y''_y)\cos(vy) + \\
&\qquad\qquad + (Y'_z - Y''_z)\cos(vz),\\
&0 = (Z'_x - Z''_x)\cos(vx) + (Z'_y - Z''_y)\cos(vy) + \\
&\qquad\qquad + (Z'_z - Z''_z)\cos(vz).
\end{aligned}\right\} \quad (5.51)$$

Proceeding from the principle of superposition, we can take the differences of stresses entering in equations (5.51) as a new system of stresses. Equations (5.51) show, however, that these stresses exist in the absence of surface and body forces and must therefore, be all zero on the basis of the hypothesis of the natural state of a body, i.e.,

$$X'_x = X''_x, \quad X_y = X''_y, \quad \text{etc.}$$

Hence, both systems of stresses (5.48′) and (5.48″) are coincident, which was to be proved.

If the principle of superposition does not hold in a given problem, we again encounter a multiple-valued solution. This can be illustrated by Euler's problem of the buckling of a bar. Indeed, let us take two loads, P_1 and P_2, each of which is slightly less than the critical P_{cr}. When subjected to each of the loads P_1 and P_2 separately the bar undergoes simple compression. Under the action of the sum of loads P_1+P_2, however, colossal bending stresses are added to the total compressive stress; the principle of superposition is not, therefore, applicable here and the action of the sum of the forces is not equal to the sum of their separate actions. At the same time the solution appears to be multiple-valued, since several configurations of equilibrium are

possible with a particular system of stresses corresponding to each of them.

The question of uniqueness of solution will be discussed in more general form in Sec. 37.

36. BELTRAMI-MICHELL EQUATIONS

Let us now turn to the general case of solving the elasticity problem in terms of stresses. It has already been pointed out more than once (Sections 24 and 29) that if stresses are chosen as basic unknowns, the number of these unknowns will be equal to six:

$$X_x,\ Y_y,\ Z_z,\ Y_z,\ Z_x,\ X_y. \tag{5.52}$$

Hence three equations of equilibrium (I)

$$\left.\begin{aligned}
&\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+\rho X=0;\\
&\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+\rho Y=0;\\
&\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+\rho Z=0
\end{aligned}\right\} \tag{I}$$

are inadequate, and to solve the problem one has also to make appeal to the compatibility conditions (IV). In addition, the conditions (II) must, of course, be satisfied on the surface:

$$\begin{aligned}
X_v&=X_x\cos(vx)+X_y\cos(vy)+X_z\cos(vz);\\
Y_v&=Y_x\cos(vx)+Y_y\cos(vy)+Y_z\cos(vz);\\
Z_v&=Z_x\cos(vx)+Z_y\cos(vy)+Z_z\cos(vz).
\end{aligned}$$

Thus the problem is reduced to the integration of nine equations (I) and (IV) with six unknown functions. Arbitrary functions entering in the general solution of these equations must be determined from the surface conditions (II).

Since the conditions of continuity (IV) connect the components of strain

$$e_{xx},\ e_{yy},\ e_{zz},\ e_{yz},\ e_{zx},\ e_{xy},$$

we have to transform these conditions expressing strains in terms of stresses (5.52) by means of Hooke's law (V). Carrying out this substitution and simultaneously using the equations of equilibrium (I), we transform equations (IV) to the following form

in the absence of body forces or if these forces are constant[1]

$$\left.\begin{aligned}
&(1+\sigma)\nabla^2 X_x+\frac{\partial^2\Theta}{\partial x^2}=0;\\
&(1+\sigma)\nabla^2 Y_y+\frac{\partial^2\Theta}{\partial y^2}=0;\\
&(1+\sigma)\nabla^2 Z_z+\frac{\partial^2\Theta}{\partial z^2}=0;\\
&(1+\sigma)\nabla^2 Y_z+\frac{\partial^2\Theta}{\partial y\,\partial z}=0;\\
&(1+\sigma)\nabla^2 Z_x+\frac{\partial^2\Theta}{\partial z\,\partial x}=0;\\
&(1+\sigma)\nabla^2 X_y+\frac{\partial^2\Theta}{\partial x\,\partial y}=0.
\end{aligned}\right\}\qquad\text{(VII)}$$

Thus to solve the problem it will be necessary to integrate nine equations (I) and (VII) and to satisfy the surface conditions (II).

E. Beltrami deduced equations (VII) in a somewhat different manner, starting with Lamé's equations (VI). The line of reasoning is then as follows. Let us suppose, as above, that there are no body forces or that they are constant at all points of a body, i.e.,

$$\frac{\partial X}{\partial x}=\frac{\partial Y}{\partial y}=\ldots=\frac{\partial Z}{\partial z}=0. \qquad (5.53)$$

First of all we prove an auxiliary proposition that the signs of Laplace's operator and partial derivative may be interchanged, for instance,

$$\frac{\partial}{\partial x}(\nabla^2 f)=\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)=$$
$$=\frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{\partial f}{\partial x}\right)=\nabla^2\left(\frac{\partial f}{\partial x}\right).$$

Thus, if t denotes any of variables x, y, z, then

$$\frac{\partial}{\partial t}(\nabla^2 f)=\nabla^2\left(\frac{\partial f}{\partial t}\right). \qquad (5.54)$$

We prove further that under conditions (5.53) the dilatation strain θ satisfies Laplace's equation

$$\nabla^2\theta=0. \qquad (5.55)$$

For that we differentiate the first of equations (VI) with respect to x, the second with respect to y and the third with respect to z;

[1] This derivation is omitted here; it may be found in S. P. Timoshenko, *Theory of Elasticity*, New York, McGraw-Hill Book Co., 1934.

adding up the obtained results and using (5.53) and (5.54), we get

$$(\lambda + 2\mu)\,\nabla^2\theta = 0,$$

whence we arrive at equation (5.55).

Now by differentiating Lamé's equations (VI), we shall derive six differential relations between strains. Differentiating the first of equations (VI) with respect to x and using equations (5.53) and (5.54), we obtain

$$(\lambda+\mu)\frac{\partial^2\theta}{\partial x^2} + \mu\nabla^2 e_{xx} = 0. \tag{5.56}$$

Hence by a cyclic change we obtain two more analogous relations:

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial^2\theta}{\partial y^2} + \mu\nabla^2 e_{yy} = 0,\\ (\lambda+\mu)\frac{\partial^2\theta}{\partial z^2} + \mu\nabla^2 e_{zz} = 0.\end{aligned}\right\} \tag{5.57}$$

We now differentiate the second of equations (VI) with respect to z and the third with respect to y, and, adding up the results, we get

$$2(\lambda+\mu)\frac{\partial^2\theta}{\partial y\,\partial z} + \mu\nabla^2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0,$$

or

$$2(\lambda+\mu)\frac{\partial^2\theta}{\partial y\,\partial z} + \mu\nabla^2 e_{yz} = 0. \tag{5.58}$$

A cyclic change yields two more relations:

$$\left.\begin{aligned}2(\lambda+\mu)\frac{\partial^2\theta}{\partial z\,\partial x} + \mu\nabla^2 e_{zx} = 0,\\ 2(\lambda+\mu)\frac{\partial^2\vartheta}{\partial x\,\partial y} + \mu\nabla^2 e_{xy} = 0.\end{aligned}\right\} \tag{5.59}$$

In our case (when there are no body forces or they are constant) six equations (5.56)-(5.59) connecting the second derivatives of strains of an isotropic elastic body are equivalent to six Saint-Venant's equations (VI) and can replace them. Now it remains to convert equations (5.56)-(5.59) to stresses on the basis of Hooke's law (V′) and (V′a) we have

$$\theta = \frac{1}{3\lambda + 2\mu}\,\Theta.$$

From (V′) we obtain

$$e_{xx} = \frac{X_x - \lambda\theta}{2\mu}$$

and further, using (5.55), we determine

$$\nabla^2 e_{xx} = \frac{1}{2\mu} \nabla^2 X_x.$$

Substituting this in (5.56), we find that

$$\frac{2(\lambda+\mu)}{3\lambda+2\mu} \frac{\partial^2 \Theta}{\partial x^2} + \nabla^2 X_x = 0. \tag{5.60}$$

But (see Sec. 18)

$$1+\sigma = 1 + \frac{\lambda}{2(\lambda+\mu)} = \frac{3\lambda+2\mu}{2(\lambda+\mu)}. \tag{5.61}$$

Keeping this in mind, we note that (5.60) coincides with the first of equations (VII). The second and third of equations (VII) will be obtained from (5.61) by a cyclic change.

Let us turn to equations (5.58)-(5.59); since, according to (V′), we have

$$e_{yz} = \frac{1}{\mu} Y_z,$$

substituting this and from (V′a) in (5.58), we obtain

$$\frac{2(\lambda+\mu)}{3\lambda+2\mu} \frac{\partial^2 \Theta}{\partial y\, \partial z} + \nabla^2 Y_z = 0,$$

which, according to (5.61), coincides with the fourth of equations (VII); the other two will be obtained from this by a cyclic change.

J. H. Michell derived equations (VII) for the general case when body forces are not constant. Then the relations become more complicated and, for instance, the first of equations (VII) is replaced by

$$\nabla^2 X_x + \frac{1}{1+\sigma} \frac{\partial^2 \Theta}{\partial x^2} = -\frac{1}{1-\sigma} \rho \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) - 2\rho \frac{\partial X}{\partial x}.$$

The reader is recommended to derive these equations on his own; for that he should repeat the derivation given in this section but retain the body forces ρX, ρY, ρZ in Lamé's equations.

*37. THREE KINDS OF PROBLEMS OF THE THEORY OF ELASTICITY. UNIQUENESS THEOREM

The Beltrami-Michell equations complete the system of equations of the theory of elasticity, permitting us to solve the necessary problems in terms of displacements or stresses.

Up to this point we have assumed that there are prescribed loads on the surface of an elastic body, as well as body forces. The problem thus formulated will be called *the first basic problem*

of the theory of elasticity. In applied elasticity another case is encountered when displacements u, v, w are prescribed on the surface of an elastic body at all its points; let us define this case as *the second basic problem of the theory of elasticity*; there also occur stresses on the surface, together with the given displacements of its points, but they are not known beforehand. *A mixed problem of the theory of elasticity* is also possible when displacements are prescribed on one part of the surface, and loads (stresses) on the other; in all these problems there may also exist body forces specified in advance.

The above three basic problems constitute a certain schematisation of real physical problems; the so-called contact problems are closer to actual conditions; these problems deal with the mutual pressure of two bodies: of two elastic bodies or of an absolutely rigid body against an elastic body; one has to deal with a contact problem, for instance, in the design of bearings (simple and ball bearings), rollers and plates, movable supports of trusses and beams as well as in the problems of pressure of a punch on a plane surface of an elastic body.

Given in Sec. 35 was the proof of the uniqueness of solution of the first basic problem of the theory of elasticity; we shall now extend it to the second and mixed problems; the proof, presented below, was given by G. Kirchhoff; it is based on the properties of the work of forces producing deformation of an elastic body.

Let X_v, Y_v, Z_v be the projections of an external load applied at the surface of a body, which are related to stresses near the surface by equations (II); let u, v, w be the displacements of the respective point of the surface. Let us set up the following double integral extended over the surface of the body:

$$J = \int_{(S)} (X_v u + Y_v v + Z_v w)\, dS.$$

For the subsequent derivation the physical meaning of this integral need not be given, but it can easily be observed that it represents the doubled work of external loads in the process of deformation of a body if these loads increase very slowly from the initial natural state of the body. This follows from Clapeyron's theorem [1].

Let us replace X_v, Y_v, Z_v by their expressions in accordance with formulas (II) and collect together the coefficients of the cosines of the angles between the outward normal v and the

[1] This theorem is expressed below in the general form by equation (5.62).

co-ordinate axes. Our integral will then become

$$J=\int_{(S)}[P\cos(\nu x)+Q\cos(\nu y)+R\cos(\nu z)]\,dS,$$

where

$$P=X_x u+Y_x v+Z_x w,$$
$$Q=X_y u+Y_y v+Z_y w,$$
$$R=X_z u+Y_z v+Z_z w.$$

On the basis of the Green-Ostrogradsky formula presented in Sec. 3 (footnote on p. 25) this integral can be transformed into an integral extended over the volume of the body τ:

$$J=\int_{(\tau)}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)d\tau.$$

We calculate the derivatives here

$$\frac{\partial P}{\partial x}=\frac{\partial X_x}{\partial x}u+\frac{\partial Y_x}{\partial x}v+\frac{\partial Z_x}{\partial x}w+X_x\frac{\partial u}{\partial x}+Y_x\frac{\partial v}{\partial x}+Z_x\frac{\partial w}{\partial x},$$
$$\frac{\partial Q}{\partial y}=\frac{\partial X_y}{\partial y}u+\frac{\partial Y_y}{\partial y}v+\frac{\partial Z_y}{\partial y}w+X_y\frac{\partial u}{\partial y}+Y_y\frac{\partial v}{\partial y}+Z_y\frac{\partial w}{\partial y},$$
$$\frac{\partial R}{\partial z}=\frac{\partial X_z}{\partial z}u+\frac{\partial Y_z}{\partial z}v+\frac{\partial Z_z}{\partial z}w+X_z\frac{\partial u}{\partial z}+Y_z\frac{\partial v}{\partial z}+Z_z\frac{\partial w}{\partial z}.$$

Let us add up these equalities by members, taking into account that in summing up the first three columns on the right-hand side we get on the basis of equations (I)

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=-\rho X, \text{ etc.}$$

In summing up the last three columns, we take into account the law of reciprocity of shearing stresses (1.6) and formulas (III); hence we obtain

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=-\rho Xu-\rho Yv-\rho Zw+$$
$$+X_x e_{xx}+Y_y e_{yy}+Z_z e_{zz}+Y_z e_{yz}+Z_x e_{zx}+X_y e_{xy},$$

or on the basis of (3.32)

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=-\rho Xu-\rho Yv-\rho Zw+2W.$$

By using this relation in the transformation of the integral J in question, we arrive at the following equality:

$$\int_{(S)}(X_\nu u+Y_\nu v+Z_\nu w)\,dS+\int_{(\tau)}(\rho Xu+\rho Yv+\rho Zw)\,d\tau=2\int_{(\tau)}W\,d\tau. \tag{5.62}$$

The first integral in the left-hand member as has already been stated, represents the doubled work of surface tractions done in the process of deformation; the second integral represents the doubled work of body forces; the right-hand member involves the doubled potential elastic energy accumulated in the body. Obviously, relationship (5.62) formulates the assumption made at the beginning of Sec. 20 in Chapter III concerning the existence of the potential of elastic forces; according to this hypothesis, the work done by surface and body forces must be completely stored in the form of elastic potential energy.

Equality (5.62) makes it easy to prove the theorem of uniqueness of solution for the three basic problems of the theory of elasticity; for this purpose let us continue to reason as in Sec. 35 and assume that we have obtained two different systems of stresses, displacements and strains, as designated in (5.48), subject to the same conditions on the surface and the same body forces.

If we accept the principle of superposition, the "difference of these solutions"

$$\left.\begin{array}{llll} X_v = X_v' - X_v''; & Y_v = Y_v' - Y_v''; & & Z_v = Z_v' - Z_v'', \\ X_x = X_x' - X_x''; & Y_y = Y_y' - Y_y''; & \ldots & X_y = X_y' - X_y'', \\ u = u' - u''; & v = v' - v''; & & w = w' - w'', \\ e_{xx} = e_{xx}' - e_{xx}''; & e_{yy} = e_{yy}' - e_{yy}''; & \ldots & e_{xy} = e_{xy}' - e_{xy}'' \end{array}\right\} \qquad (5.63)$$

can also be taken as a solution of a certain problem of the theory of elasticity; consequently, one may apply equality (5.62) to this solution in its simplified form with the second integral in the left-hand member vanishing as the body forces are identical in both solutions and, therefore,

$$\rho X = \rho X' - \rho X'' = 0,$$

etc.

Thus, we obtain

$$\iint\limits_{(S)} (X_v u + Y_v v + Z_v w)\, dS = 2 \iiint\limits_{(\tau)} W\, d\tau, \qquad (5.64)$$

where X_v, Y_v, Z_v, u, v, w are taken from the first and third lines of (5.63); if the function W is represented by (3.33), its arguments should be taken from the fourth line of (5.63); if, however, it is given in the form (3.34), its arguments should be taken from the second line of (5.63). Turning now to the integral in the left-hand member of (5.64), we note that

(1) in the case of the first basic problem we shall have everywhere on the surface

$$X_v = 0, \quad Y_v = 0, \quad Z_v = 0; \tag{5.65}$$

(2) in the case of the second basic problem we shall have everywhere on the surface

$$u = 0, \quad v = 0, \quad w = 0; \tag{5.66}$$

(3) in the case of the mixed problem conditions (5.65) are satisfied on the portions of the surface where stresses are given, while on the remainder where displacements are given conditions (5.66) are fulfilled.

It is obvious that in all three cases the integral in the left-hand member of (5.64) vanishes and there results the equality

$$\int\limits_{(\tau)} W\, d\tau = 0.$$

At the end of Sec. 21, however, it was shown that

$$W > 0$$

at all points of the body; the last equality, therefore, is possible only if all arguments of the function W are zero, i.e.,

$$e_{xx} = e'_{xx} - e''_{xx} = 0, \quad \text{etc.}$$

$$X'_x - X''_x = 0, \quad \text{etc.}$$

Hence, it follows that both systems of stresses assumed at the start must coincide at all points of the body; the same applies to strains as well.

This proves the uniqueness theorem. It does not follow, however, from the foregoing proof that displacements will also be identical in both of the assumed solutions; in the case of the first basic problem, where stresses are prescribed on the surface, this will not be true; in fact, from the conditions

$$e_{xx} = e'_{xx} - e''_{xx} = 0, \quad \text{etc.,}$$

according to what has been said in Sec. 10 concerning formulas (2.11), it follows that one may give any small additional rigid body displacement to an elastic body having certain stresses and strains under the action of given forces. Obviously, in the second and mixed problems displacements will also be the same in both solutions, since they are specified over the whole or a part of the surface of a body.

VI

Plane Problem in Cartesian Co-ordinates

38. PLANE STRAIN

We now proceed to a large category of problems of the theory of elasticity which are important for practical application and at the same time admit considerable simplification in the mathematical aspect of solution.

Simplification implies that one may disregard one of the co-ordinate axes in these problems, for instance Oz, and consider that the whole phenomenon takes place in one plane Oxy. It is very difficult to realise such a case in pure form in practice, but we encounter something like it in many problems. These problems fall into two groups opposite in a sense but common by the mathematical form of solution.

The first of these groups corresponds to the case when one of the displacements, for instance w, is zero everywhere, while the other two, u and v, are not dependent on the co-ordinate z corresponding to the displacement w; this case is thus characterised by the following conditions which are valid throughout the body:

$$\left.\begin{aligned} &u = f_1(x,\ y);\quad v = f_2(x,\ y);\\ &w = 0. \end{aligned}\right\} \tag{6.1}$$

In these conditions equations (III) give

$$\left.\begin{aligned} &e_{zz} = 0;\ e_{yz} = 0;\quad e_{zx} = 0;\\ &e_{xx} = \varphi_1(x,\ y);\qquad e_{yy} = \varphi_2(x,\ y);\\ &e_{xy} = \varphi_3(x,\ y). \end{aligned}\right\} \tag{6.2}$$

Equations (6.1) and (6.2) show that all displacements and deformations take place exclusively in the directions parallel to the plane Oxy, the pattern of displacements and strains being the same in all sections of the body parallel to the plane Oxy

($z=z_0$, where z_0 is any number). Such deformation is called plane strain.

A case similar to this is encountered in problems dealing with a long prismatical or cylindrical body with the axis parallel to the axis Oz, under a lateral load normal to the axis Oz and constant along the axis (though perhaps varying in the directions normal to the axis). Such, for instance, are the problems of a long dam (Fig. 46*a*), a long roller (Fig. 46*b*), a long dome (Fig. 46*c*), a long plate (Fig. 46*d*) with an axis parallel to the

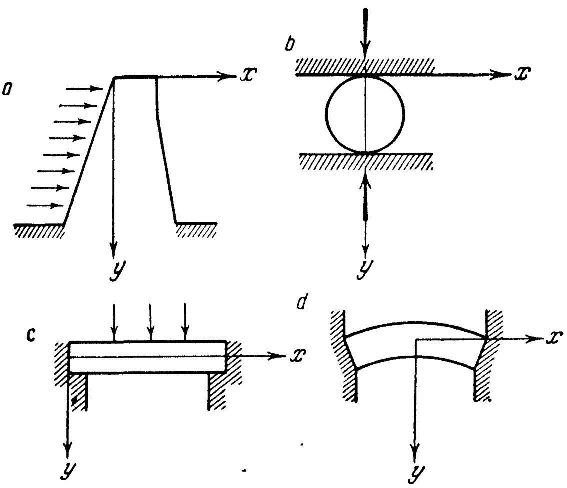

Fig. 46

axis Oz; let us suppose that the load does not vary along the axis Oz in all these cases.

If we isolate a thin element with a load acting upon it in such a long prismatical body far from its ends by two sections a short distance apart and parallel to the plane Oxy and imagine that it works as a separate elastic body, we note that there must occur elongations in it along the axis Oz; these elongations result from the transverse action of the load parallel to the plane Oxy. Actually, however, this element is adjacent to two other elements having the same elongation but in the opposite direction; as a result of the interaction of these elements there will be neither elongations nor displacements along the axis Oz, but there will occur forces of interaction between the adjacent elements represented by normal stresses Z_z developed due to the elimination of the strain e_{zz}.

Indeed, take the third of the equations of Hooke's law (V); the elimination of the strain e_{zz} will lead to the equation

$$Z_z - \sigma(X_x + Y_y) = 0,$$

or

$$Z_z = \sigma(X_x + Y_y). \tag{6.3}$$

Consequently, $Z_z' \neq 0$, but it is a function of primary stresses X_x and Y_y produced by the load.

These considerations will now be illustrated by an example. Let us have a blind flooring composed of a number of rectangular beams subjected to bending (Fig. 47*a*); let the axis Ox be

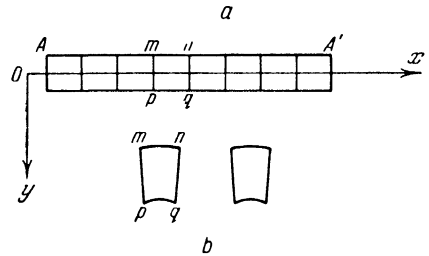

Fig. 47

directed along their span. The cross sections of the beams distort in bending (Fig. 47*b*), as has been shown previously (Sec. 33), and there result elongations and contractions e_{zz}. If, however, we have a solid wide slab of width AA' in place of a beam flooring, no distortion of its individual elements *mnpq* isolated far from the ends A and A' can occur under flexure; there will appear stresses Z_z instead, since at the top points n (Fig. 47*b*) the elements will press on one another while those at the bottom points q will stretch. These stresses Z_z are expressed by formula (6.3). Furthermore, assuming that the neighbouring elements of the prismatical body under consideration (Figs 46 and 47) carry the same load and work in exactly the same conditions, there is no reason to expect any shear deformations between them; hence we can easily see that everywhere $e_{yz}=0$, $e_{zx}=0$. Thus conditions (6.1), (6.2), (6.3) of plane strain are satisfied approximately for the elements sufficiently remote from the ends of the long prismatical body under consideration. One can easily see that in these conditions all our basic groups of equations are considerably simplified. Let us begin with Hooke's law in the form (V); the first of these equations on the basis of (6.3)

becomes

$$e_{xx}=\frac{1}{E}[X_x-\sigma Y_y-\sigma^2(X_x+Y_y)]=\frac{1+\sigma}{E}[(1-\sigma)X_x-\sigma Y_y],$$

the second equation becomes

$$e_{yy}=\frac{1+\sigma}{E}[(1-\sigma)Y_y-\sigma X_x].$$

These equations may conveniently be put in the form

$$\left.\begin{aligned} e_{xx}&=\frac{1}{E_1}(X_x-\sigma_1 Y_y),\\ e_{yy}&=\frac{1}{E_1}(Y_y-\sigma_1 X_x),\end{aligned}\right\}\tag{6.4}$$

where

$$E_1=\frac{E}{1-\sigma^2};\quad \sigma_1=\frac{\sigma}{1-\sigma}.\tag{6.5}$$

The fourth and fifth equations on the basis of (6.2) give

$$Y_z=0;\quad Z_x=0.\tag{6.6}$$

The sixth equation becomes

$$e_{xy}=\frac{2(1+\sigma)}{E}X_y.$$

But, using (6.5), we find

$$\frac{2(1+\sigma_1)}{E_1}=\frac{2(1+\sigma)}{E},$$

and therefore

$$e_{xy}=\frac{2(1+\sigma_1)}{E_1}X_y.\tag{6.4$'$}$$

Furthermore, on the basis of (6.2), equations (6.4) show that

$$\left.\begin{aligned} &X_x=\psi_1(x,\ y),\ Y_y=\psi_2(x,\ y),\\ &X_y=Y_x=\psi_3(x,\ y),\\ &X_z=Y_z=0;\end{aligned}\right\}\tag{6.7}$$

on this basis according to equation (6.3)

$$Z_z=\psi_4(x,\ y),$$

i.e., all nonvanishing stresses are independent of the co-ordinate z, which is, of course, clear all along.

The following conclusions can be drawn from the above. There remain only the first two equations of the group of equilibrium

equations (I); these become

$$\left.\begin{aligned} \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\rho X=0; \\ \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\rho Y=0. \end{aligned}\right\} \qquad \text{(Ip)}$$

The surface conditions (II) are simplified as follows:

$$\left.\begin{aligned} X_v=X_x\cos(vx)+X_y\cos(vy); \\ Y_v=Y_x\cos(vx)+Y_y\cos(vy). \end{aligned}\right\} \qquad \text{(IIp)}$$

Relations (III) reduce to

$$\left.\begin{aligned} e_{xx}=\frac{\partial u}{\partial x}; \quad e_{yy}=\frac{\partial v}{\partial y}; \\ e_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}. \end{aligned}\right\} \qquad \text{(IIIp)}$$

As can easily be seen, there remains only the first of the six conditions of continuity (IV)

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}-\frac{\partial^2 e_{xy}}{\partial x\,\partial y}=0. \qquad \text{(IVp)}$$

Hooke's law (V) becomes

$$\left.\begin{aligned} e_{xx}&=\frac{1}{E_1}(X_x-\sigma_1 Y_y); \\ e_{yy}&=\frac{1}{E_1}(Y_y-\sigma_1 X_x); \\ e_{xy}&=\frac{2(1+\sigma_1)}{E_1}X_y. \end{aligned}\right\} \qquad \text{(Vp)}$$

Equations (Ip)-(Vp) make it possible to proceed to the solution of the problem of plane strain.

39. GENERALISED PLANE STRESS. MAURICE LÉVY'S EQUATION. STRESS FUNCTION

Let us now turn to the other case, which is analogous to the preceding one but opposite in the sense of extension along the axis Oz, i.e., let us consider again the problems illustrated in Fig. 46. Suppose, however, that the length of the prismatical body along the axis Oz is very small; we shall thus have a thin plate (of width h) loaded over the lateral surface (edge) by forces parallel to its faces. The equilibrium of the plate, so long as the load does not exceed a certain limit (critical value),

will be stable and the plate will not deflect in the direction of the axis Oz. This case is called the *generalised plane stress.*

The faces of the plate (parallel to the plane Oxy) are assumed to be free of stress, i.e., everywhere on them

$$Z_z = 0; \quad X_z = 0; \quad Y_z = 0.$$

But since the distance between these faces (the width of the plate) is very small, the stresses can be considered to be vanishingly small within the plate as well [1]. The remaining stress components X_x, Y_y, X_y will vary very slightly along the axis Oz for the same reason (small thickness along the axis Oz) and, therefore, we shall assume them to be independent of the co-ordinate z. Thus in the problem under consideration the following conditions are satisfied approximately for stresses:

$$\left.\begin{aligned} X_x &= \psi_1(x, y); & Y_y &= \psi(x, y); \\ X_y &= \psi_3(x, y); & Y_z &= Z_y = 0; \\ Z_x &= X_z = 0; & Z_z &= 0. \end{aligned}\right\} \tag{6.8}$$

They differ from the respective conditions (6.6) and (6.7) for plane strain only by the condition $Z_z=0$.

As for strains, this problem differs from the problem of plane strain by the fact that here e_{zz} is not zero [cf. formulas (6.2)]. It will represent the transverse deformation along the axis Oz, produced by primary stresses X_x, Y_y lying in the planes parallel to Oxy.

The strain e_{zz} will entail some distortion of the plane faces of the plate. In view of the small thickness of the plate, however, this distortion will be very slight. Let us take, for instance, the case of the pure bending of a rectangular beam considered in Sec. 33. The load-free lateral edges of the cross section incline through an angle γ (Fig. 44b); one can see, however, from formula (5.31) of Sec. 33 that this angle is proportional to the width of the beam b; therefore, if b is small, i.e., if the beam has the shape of a thin plate, angles γ will also be very small and the distortion of its lateral surfaces will be negligible.

Proceeding from equations (6.8), we draw the following conclusions: the basic groups of equations (I) and (II) have the same form (Ip) and (IIp) in this case as for plane strain; it is also

[1] Such an assumption is justifiable with regard to normal stresses Z_z; as to stresses X_z and Y_z, the assumption is true if the load on the lateral surface is distributed uniformly along the generator (as was supposed in the case of plane strain). Let us generalise the problem and assume that the load is distributed symmetrically with respect to the middle plane of the plate O—O, but nonuniformly (Fig. 48a); then shearing stresses Z_x or Z_y may

sufficient to retain only equations (IIIp) of group (III); equally, we may limit ourselves to one equation (IVp) in group (IV); as regards Hooke's law (V), on the basis of the last of conditions (6.8), there will be a distinction from equations (Vp) of the problem of plane strain; indeed, in our case we have

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E}(X_x - \sigma Y_y), \\ e_{yy} &= \frac{1}{E}(Y_y - \sigma X_x), \\ e_{xy} &= \frac{2(1+\sigma)}{E} X_y. \end{aligned}\right\} \qquad \text{(V'p)}$$

Summarising the foregoing considerations we conclude that in the solution of problem's both of plane strain and generalised plane stress, we may use the basic groups of equations: (Ip),

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be considerable in the section *mn*; however, the curve of their distribution (diagram) will be antisymmetrical and its area will be equal to zero; this can be expressed in the form of equalities

$$\frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} Z_x\,dz = 0; \quad \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} Z_y\,dz = 0.$$

It may thus be said that shearing stresses Z_x and Z_y will be *equal to zero on the average across the thickness of the plate.* Therefore, in a more strict presentation of the theory of generalised plane stress all the components of

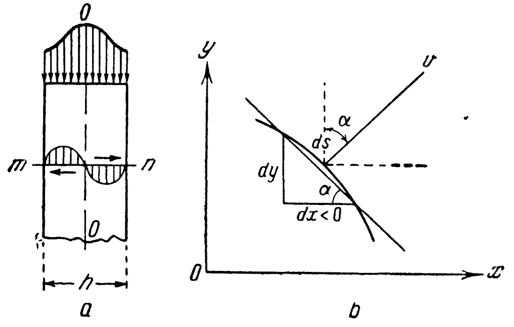

Fig. 48

the stress tensor are replaced by their average values across the thickness of the plate. For simplicity we shall assume in future that surface loads are distributed uniformly along generators and, therefore, we shall assume that throughout the plate

$$Z_z = X_z = Z_x = Y_z = Z_y = 0.$$

(IIp), (IIIp) and (IVp). Hooke's law, however, has different expressions for these problems: equations (Vp) for plane strain, and equations (V′p) for plane stress. It is important, however, to note that the form of these equations is the same in both cases; the difference is only in the values of the elastic constants which are expressed in terms of E and σ by formulas (6.5) in the case of plane strain.

Henceforth we choose stresses X_x, Y_y, $X_y = Y_x$ as unknowns; it is, therefore, necessary to transform the equation of compatibility (IVp) replacing strains by their expressions from equations (Vp) or (V′p), depending on whether we have plane strain or plane stress.

Let us differentiate equations (Vp) as required by equation (IVp)

$$\left.\begin{aligned} \frac{\partial^2 e_{xx}}{\partial y^2} &= \frac{1}{E_1}\left(\frac{\partial^2 X_x}{\partial y^2} - \sigma_1 \frac{\partial^2 Y_y}{\partial y^2}\right), \\ \frac{\partial^2 e_{yy}}{\partial x^2} &= \frac{1}{E_1}\left(\frac{\partial^2 Y_y}{\partial x^2} - \sigma_1 \frac{\partial^2 X_x}{\partial x^2}\right), \end{aligned}\right\} \tag{6.9}$$

$$\frac{\partial^2 e_{xy}}{\partial x\,\partial y} = \frac{2(1+\sigma_1)}{E_1}\,\frac{\partial^2 X_y}{\partial x\,\partial y}. \tag{6.10}$$

We transform the right-hand member of equation (6.10) expressing shearing stress X_y in terms of normal stresses X_x and Y_y from equations (Ip); differentiating them and transposing some terms to the right-hand side, we have

$$\begin{aligned} \frac{\partial^2 X_y}{\partial y\,\partial x} &= -\frac{\partial^2 X_x}{\partial x^2} - \rho\frac{\partial X}{\partial x}, \\ \frac{\partial^2 Y_x}{\partial x\,\partial y} &= -\frac{\partial^2 Y_y}{\partial y^2} - \rho\frac{\partial Y}{\partial y}. \end{aligned}$$

Adding up by members, we obtain

$$2\frac{\partial^2 X_y}{\partial x\,\partial y} = -\frac{\partial^2 X_x}{\partial x^2} - \frac{\partial^2 Y_y}{\partial y^2} - \rho\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right). \tag{6.11}$$

In the sequel we shall restrict ourselves to the case when body forces are constant throughout the volume of the body; then

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = 0.$$

In particular, we shall deal with the gravity force and, if we direct the axis Oy downward, we shall have

$$\left.\begin{aligned} X = 0,\ Y &= g, \\ \rho Y = \rho g &= p, \end{aligned}\right\} \tag{6.12}$$

where p is the weight per unit volume of the body; now equation (6.11) becomes

$$2\frac{\partial^2 X_y}{\partial x\,\partial y}=-\left(\frac{\partial^2 X_x}{\partial x^2}+\frac{\partial^2 Y_y}{\partial y^2}\right),$$

and equations (6.9) and (6.10) are rewritten as

$$\frac{\partial^2 e_{xx}}{\partial y^2}=\frac{1}{E_1}\left(\frac{\partial^2 X_x}{\partial y^2}-\sigma_1\frac{\partial^2 Y_y}{\partial y^2}\right),$$

$$\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{1}{E_1}\left(\frac{\partial^2 Y_y}{\partial x^2}-\sigma_1\frac{\partial^2 X_x}{\partial x^2}\right),$$

$$-\frac{\partial^2 e_{xy}}{\partial x\,\partial y}=\frac{1+\sigma_1}{E_1}\left(\frac{\partial^2 X_x}{\partial x^2}+\frac{\partial^2 Y_y}{\partial y^2}\right).$$

Adding them up by members, we obtain the left-hand member of (IVp) in the form converted to stresses:

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}-\frac{\partial^2 e_{xy}}{\partial x\,\partial y}=\frac{1}{E_1}\left[\frac{\partial^2 X_x}{\partial x^2}+\frac{\partial^2 Y_y}{\partial x^2}+\frac{\partial^2 X_x}{\partial y^2}+\frac{\partial^2 Y_y}{\partial y^2}\right]=$$

$$=\frac{1}{E_1}\left[\frac{\partial^2 (X_x+Y_y)}{\partial x^2}+\frac{\partial^2 (X_x+Y_y)}{\partial y^2}\right].$$

Here the bracketed expression represents the action of the Laplacian operator of the second order on the function (X_x+Y_y):

$$\nabla^2(X_x+Y_y)=\frac{\partial^2 (X_x+Y_y)}{\partial x^2}+\frac{\partial^2 (X_x+Y_y)}{\partial y^2}.$$

Introducing this in the equation of compatibility (IVp), we obtain it in the final form, converted to stresses:

$$\nabla^2(X_x+Y_y)=0.$$

This is the so-called M. Lévy's condition. It may be obtained in a different way if we proceed from the fact that in the absence of body forces the volume expansion θ is a harmonic function [cf. formula (5.55)]:

$$\nabla^2\theta=0,$$

and, consequently, according to (3.10) we have

$$\nabla^2\Theta=0,\ \text{where}\ \Theta=X_x+Y_y+Z_z.$$

But in the case of plane strain

$$Z_z=\sigma(X_x+Y_y),$$

and in the case of generalised plane stress

$$Z_z=0.$$

Hence we obtain

$$\nabla^2 (X_x + Y_y) = 0.$$

Thus the solution of the plane problem is reduced to integration of three differential equations:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0; \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + p = 0; \tag{Ip}$$

$$\nabla^2 (X_x + Y_y) = 0, \tag{IV'p}$$

and satisfaction of the conditions on the surface:

$$\left. \begin{aligned} X_v &= X_x \cos (vx) + X_y \cos (vy), \\ Y_v &= Y_x \cos (vx) + Y_y \cos (vy). \end{aligned} \right\} \tag{IIp}$$

If in this way we find stresses

$$X_x, \; Y_y, \; X_y, \tag{6.13}$$

then, introducing them in (Vp) for plane strain or in (V'p) for plane stress, we determine strains e_{xx}, e_{yy}, e_{xy} as functions of x and y; substituting them further in equations (IIIp), we get

$$\frac{\partial u}{\partial x} = e_{xx}; \quad \frac{\partial v}{\partial y} = e_{yy}; \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = e_{xy}, \tag{IIIp}$$

and, integrating the last equations, we obtain displacements u, v. This completes the solution of the problem.

It is very important to note the following. If it is only stresses that are to be found in a given problem, one may confine oneself to equations (Ip), (IV'p) and (IIp) which do not involve the elastic constants. Hence one might expect that the state of stress does not depend on the material; this conclusion is true for a body bounded by a simply connected surface (which is represented by a simply connected contour in the plane of action of forces). In the case of a multiply connected contour, the state of stress does not depend on the material if the external loads are balanced at each of the contours separately [1]. These conclusions underlie the practical application of the photoelastic method of stress analysis in polarised light; a given material is replaced then by another one—a transparent and photoelastic material from which a plate is made and tested as a model.

G. B. Airy indicated the possibility of further simplification in the solution of the problem. This simplification is based on the fact that we can easily find the general solution of the system of equations (Ip) so as not to deal with them later. The system (Ip)

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[1] To be exact, if the resultant of the loads is equal to zero at each contour separately.

is nonhomogeneous and, therefore, its general solution represents the sum of the general solution of the homogeneous system

$$\left.\begin{aligned} \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \\ \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0 \end{aligned}\right\} \tag{6.14}$$

and a particular solution of the system (Ip); this particular solution can easily be found if we assume, for instance [see formula (6.12)],

$$X_x=Y_y=0; \quad X_y=-px, \tag{6.15}$$

or

$$X_x=X_y=0; \quad Y_y=-py. \tag{6.15'}$$

Let us now turn to system (6.14). Its general solution contains one arbitrary function $\varphi(x, y)$ of the independent variables x and y and has a simple form

$$X_x=\frac{\partial^2\varphi}{\partial y^2}; \quad Y_y=\frac{\partial^2\varphi}{\partial x^2}; \quad X_y=Y_x=-\frac{\partial^2\varphi}{\partial x\,\partial y}. \tag{6.16}$$

If we substitute these expressions of stresses in equations (6.14), we shall see that they are satisfied identically, whatever function $\varphi(x, y)$ may be, provided its partial derivatives up to the fourth order inclusive exist and are continuous. This function is called the *stress function* or *Airy's function*.

The general solution (6.16) can readily be obtained as follows: we satisfy the first equation of the system (6.14), assuming

$$X_x=\frac{\partial\psi}{\partial y}; \quad X_y=-\frac{\partial\psi}{\partial x}, \tag{6.17}$$

where $\psi(x, y)$ is an arbitrary function. The second equation will be satisfied if

$$X_y=\frac{\partial\chi}{\partial y}, \quad Y_y=-\frac{\partial\chi}{\partial x}, \tag{6.18}$$

where $\chi(x, y)$ is also an arbitrary function. Both equations (6.14), however, will be satisfied simultaneously only if the expressions of X_y in (6.16) and (6.17) coincide, so as to have

$$-\frac{\partial\psi}{\partial x}=\frac{\partial\chi}{\partial y} \text{ or } \frac{\partial\psi}{\partial x}+\frac{\partial\chi}{\partial y}=0;$$

this equation will be satisfied if we assume

$$\psi=\frac{\partial\varphi}{\partial y}; \quad \chi=-\frac{\partial\varphi}{\partial x};$$

introducing these values of ψ and χ in (6.17) and (6.18), we obtain formulas (6.16). The general solution of equations (Ip) will be found by adding their partial solution, for instance, in the form (6.15′) to (6.16); we thus have finally

$$X_x=\frac{\partial^2\varphi}{\partial y^2};\quad Y_y=\frac{\partial^2\varphi}{\partial x^2};\quad X_y=Y_x=-\frac{\partial^2\varphi}{\partial x\,\partial y}-px. \qquad \text{(VIII)}$$

Substituting these expressions in equations (Ip), we can easily see that the latter will be satisfied identically and we shall not have to deal with them in future. There remains one equation (IV′p) which is expressible in terms of the function φ; indeed, on the basis of (VIII)

$$X_x+Y_y=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\nabla^2\varphi,$$

and equation (IV′p) will be represented symbolically as

$$\nabla^2(\nabla^2\varphi)=\nabla^2\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)=0. \qquad \text{(IX′)}$$

Writing out the symbol of Laplace's operator, we get

$$\frac{\partial^4\varphi}{\partial x^4}+2\,\frac{\partial^4\varphi}{\partial x^2\,\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0. \qquad \text{(IX)}$$

Thus the solution of the plane problem in terms of stresses is reduced to the integration of one partial differential equation (IX) of the fourth order; if we determine the function $\varphi(x, y)$ from this equation, we shall find stresses at any point of a body by formulas (6.16) or (VIII). Boundary conditions corresponding to each specific problem must, of course, be added to equation (IX). Here we shall dwell on *the first basic problem of the theory of elasticity* (Sec. 37); in this case we must prescribe stresses, i.e., loads, on the boundary of a body, i.e., on the contour of the cross section (if plane strain is considered) or on the contour of a plate (in a state of generalised plane stress). The boundary conditions will have the simplest form, if the contour of a body is a rectangle with the sides parallel to the co-ordinate axes Ox and Oy. We must then prescribe stresses Y_y and X_y on the sides parallel to the axis Ox, and stresses X_x and Y_x on the sides parallel to the axis Oy; according to (6.16), these stresses are equal to the corresponding values of the second derivatives of the function $\varphi(x, y)$ which have to be specified. Several problems of this kind are treated in the following sections.

* Consider the boundary conditions in the case of a contour of arbitrary configuration; for this purpose turn to the general conditions on the surface (IIp); by the use of the stress function

$\varphi(x, y)$ they will be written down as

$$\left.\begin{aligned} X_v &= \frac{\partial^2\varphi}{\partial y^2}\cos(vx) - \left(\frac{\partial^2\varphi}{\partial x\,\partial y} + px\right)\cos(vy), \\ Y_v &= -\left(\frac{\partial^2\varphi}{\partial x\,\partial y} + px\right)\cos(vx) + \frac{\partial^2\varphi}{\partial x^2}\cos(vy). \end{aligned}\right\} \tag{X}$$

Thus the solution of the plane problem is as follows: one must find a function $\varphi(x, y)$ satisfying equation (IX) at all points of the cross section of the body in question (Fig. 46), and equations (X) on the contour of this section, where X_v and Y_v are the projections of the external load on the co-ordinate axes. Once the function φ is found, we determine stresses by equations (VIII); further determination of strains and displacements is carried out according to equations (Vp) or (V′p) and (IIIp), as indicated above.

Remark. If there are no body forces (i.e., if $p=0$), the conditions on the surface (X) can be written in a more compact form, introducing the concept of the derivative of a function with respect to arc length of the contour of the cross section. Indeed, we have (Fig. 48*b*)

$$\cos(vx) = \sin\alpha = \frac{dy}{ds}, \quad \cos(vy) = \cos\alpha = -\frac{dx}{ds}.$$

Introducing this in equations (X), we have

$$\left.\begin{aligned} X_v &= \frac{\partial^2\varphi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2\varphi}{\partial x\,\partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial\varphi}{\partial y}\right), \\ Y_v &= -\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{dy}{ds} - \frac{\partial^2\varphi}{\partial x^2}\frac{dx}{ds} = -\frac{d}{ds}\left(\frac{\partial\varphi}{\partial x}\right). \end{aligned}\right\} \tag{6.19}$$

These equalities permit us to represent the conditions on the contour in the alternative form which is of interest for the general statement of the question concerning the integration of the equation of the plane problem (IX) with loads prescribed on the contour. We multiply (6.19) by ds and integrate with respect to s along the contour starting from an arbitrary point S_0 taken as the origin of arcs:

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x} &= -A - \int_0^s Y_v\,ds = -Y^{(s)}, \\ \frac{\partial\varphi}{\partial y} &= B + \int_0^s X_v\,ds = X^{(s)}. \end{aligned}\right\} \tag{6.20}$$

Here A and B are arbitrary constants; they express the values of the derivatives

$$\frac{\partial\varphi}{\partial x}, \quad \frac{\partial\varphi}{\partial y}$$

at the point S_0 of the contour.

To make further computations more clear, let us introduce the following analogy: we replace the contour of the body under

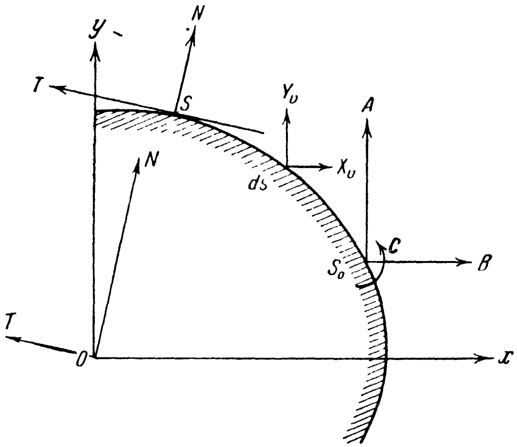

Fig. 49

investigation by a bar of the same form, which is cut at the point S_0 (Fig. 49), where we apply the forces

A parallel to the y axis,
B parallel to the x axis.

Keeping in mind that the surface loads X_v and Y_v act per unit arc length of the contour, we note that the quantities $X^{(s)}$ and $Y^{(s)}$ in the right-hand members of equalities (6.20) represent the sums of the projections of forces applied to the part S_0S of the bar on the axes Ox and Oy. If we take new axes ON and OT (Fig. 49) directed parallel to the normal and the tangent to the contour at the point S in place of the axes Ox and Oy, formulas (6.20) in the new co-ordinates n and t will be written as

$$\frac{\partial\varphi}{\partial n} = -T^{(s)}, \tag{6.21}$$

$$\frac{\partial\varphi}{\partial t} = N^{(s)}, \tag{6.22}$$

where $T^{(s)}$ is the longitudinal force at the point S of the bar and $N^{(s)}$ is the transverse force at the point S of the bar.

In formula (6.21) the quantity

$$\frac{\partial\varphi}{\partial n}$$

represents the derivative of the stress function φ along the normal to the contour; similarly

$$\frac{\partial\varphi}{\partial t}$$

is the derivative along the tangent to the contour or with respect to arc length of the contour

$$\frac{\partial\varphi}{\partial t}=\frac{\partial\varphi}{\partial s}.$$

Comparing equality (6.22) with the well known theorem on the derivative of the bending moment in a bar

$$\frac{\partial M}{\partial t}=Q,$$

we may consider that

$$\varphi=M^{(s)}, \tag{6.23}$$

where $M^{(s)}$ is the moment of the forces applied to the part S_0S of the bar with respect to the point S; in calculating $M^{(s)}$ by integration of equation (6.22) an arbitrary constant will be added which can be prescribed by applying a couple with any moment C at the initial point S_0 (Fig. 49).

All the foregoing considerations enable us to present the conditions on the contour (when loads are prescribed on it) in the required new form; for that we must calculate the values of the stress function $\varphi(x, y)$ and its normal derivative

$$\frac{\partial\varphi}{\partial n}$$

at each point of the contour on the basis of the loads prescribed on the contour, according to formulas (6.21) and (6.23) as the longitudinal force and the bending moment due to the loads prescribed on the contour. The expressions of φ and $\frac{\partial\varphi}{\partial n}$ will involve three arbitrary constants A, B and C (initial parameters); they do not, obviously, affect the prescribed contour loads X_v and Y_v and the stresses produced by them.

Functions satisfying equation (IX) are called biharmonic; the plane problem of the theory of elasticity, when loads are prescribed on the contour, can now be treated in the following math-

ematical form: it is required to find a biharmonic function

$$\varphi(x, y)$$

at all points of the region bounded by the given contour, if the values of the function φ itself and its normal derivative are prescribed on the contour.

40. SOLUTION OF THE PLANE PROBLEM BY MEANS OF POLYNOMIALS

In many problems it is often found convenient to employ the semi-inverse method specifying in advance the analytic form of a stress function $\varphi(x, y)$ and selecting its parameters (for instance, coefficients) so as to satisfy the surface conditions (X) and the basic equation (IX). Let us consider several problems in which $\varphi(x, y)$ can be prescribed in the form of an integral function (polynomial). If the function φ represents a polynomial of the second degree

$$\varphi(x, y) = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2, \tag{6.24}$$

equation (IX) will obviously be satisfied everywhere for any values of a, b and c. The stresses, in accordance with equations (VIII), will be expressed as [1]

$$X_x = c; \quad Y_y = a; \quad X_y = -b - px. \tag{6.25}$$

If $p=0$, the stresses are constant; we have the case of the *homogeneous state of stress*.

If φ is a polynomial of the third degree

$$\varphi(x, y) = \frac{d}{6}x^3 + \frac{e}{2}x^2y + \frac{f}{2}xy^2 + \frac{k}{6}y^3 + \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2, \tag{6.26}$$

equation (IX) will again be satisfied for arbitrary values of the coefficients; the stresses, according to equations (VIII), will be represented as

$$\left.\begin{aligned} X_x &= fx + ky + c, \\ Y_y &= dx + ey + a, \\ X_y &= -ex - fy - b - px, \end{aligned}\right\} \tag{6.27}$$

i.e., they will be linear functions of the co-ordinates.

[1] From equations (6.25) one can see that terms of the first degree need not be included in function (6.24) as they will not affect the magnitude of stresses.

If φ is given as a function of the fourth or higher degree, its derivatives entering in equation (IX) will be different from zero, in general; the coefficients must, therefore, be chosen in such a way as to satisfy the compatibility condition (IX) for arbitrary

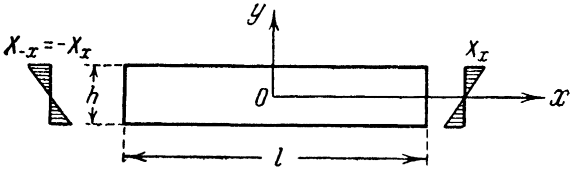

Fig. 50

values of x and y. The stresses will then be functions of the second and higher degree.

Pure bending. Consider the particular case of the stress function (6.26):

$$\varphi=\frac{k}{6}y^3.$$

Neglecting the influence of the own weight, we obtain the following stresses from equations (6.27):

$$X_x=ky;\quad Y_y=0;\quad X_y=Y_x=0. \tag{6.28}$$

If we take a plate of width h (Fig. 50) and assume that stresses (6.28) are set up in it, the following stresses will act on its surface. On the upper and lower faces $\left(\text{for } y=\pm\frac{h}{2}\right)$

$$Y_y=Y_{-y}=X_y=X_{-y}=0;$$

on the lateral faces $\left(\text{for } x=\pm\frac{l}{2}\right)$

$$X_x=ky;\quad X_{-x}=-X_x=-ky;\quad Y_x=Y_{-x}=0.$$

The diagrams of the stresses X_x are shown in the drawing; obviously, we have the case of pure bending investigated in the general form in Sec. 33.

41. BENDING OF A CANTILEVER

Consider a plate built-in at the left-hand end and subjected to a load Q distributed over the right-hand end section (Fig. 51). The elementary theory of bending gives the following system of

stresses for this case:

$$\left.\begin{aligned} X_x &= \frac{M}{J}\,y = -\frac{Q\,(l-x)}{J}\,y, \\ Y_y &= 0, \\ X_y &= Y_x = \frac{QS}{Jb} = \frac{Q\left(\frac{h^2}{4}-y^2\right)}{2J}. \end{aligned}\right\} \qquad (6.29)$$

Here M is the bending moment, Q is the shearing force, J is the moment of inertia of the cross section; in the case of generalised plane stress $J=\frac{\delta h^3}{12}$; in the case of plane strain $J=\frac{1h^3}{12}$; δ is the thickness of the plate in the direction perpendicular to the plane of the drawing.

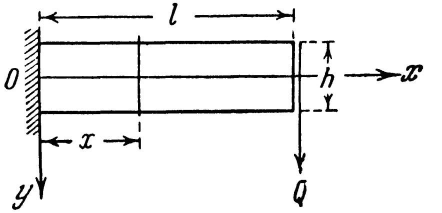

Fig. 51

Let us check whether stresses (6.29) are possible from the viewpoint of the plane problem and whether they correspond to the case of bending represented in Fig. 51. To answer the first question it is sufficient to ascertain whether stresses (6.29) can be obtained from some stress function $\varphi(x, y)$ satisfying equation (IX).

The general form of functions (6.29) expressing stresses is

$$\left.\begin{aligned} X_x &= Ay + Bxy = \frac{\partial^2\varphi}{\partial y^2}, \\ Y_y &= 0 = \frac{\partial^2\varphi}{\partial x^2}, \\ X_y &= C + Dy^2 = -\frac{\partial^2\varphi}{\partial x\,\partial y}. \end{aligned}\right\} \qquad (6.30)$$

Integrating the first of these equations twice, we have

$$\varphi(x, y) = \frac{Ay^3}{6} + \frac{Bxy^3}{6} + \psi(x)\,y + \omega(x). \qquad (6.31)$$

Differentiating this function with respect to x and substituting the results in the last two of equations (6.30), we obtain

$$\psi''(x)\,y + \omega''(x) = 0, \tag{6.32}$$

$$-\frac{By^2}{2} - \psi'(x) = C + Dy^2. \tag{6.33}$$

Both of these conditions should be satisfied for any values of x and y; therefore, we draw the conclusions:

$$\psi''(x) = 0; \quad \omega''(x) = 0;$$
$$\psi(x) = Ex + F;$$
$$\omega(x) = Hx + K;$$

substituting this in equation (6.33), we have

$$-\frac{By^2}{2} - E = C + Dy^2.$$

Hence

$$E = -C; \quad D = -\frac{B}{2}.$$

The stress function (6.31) becomes

$$\varphi(x, y) = \frac{Ay^3}{6} + \frac{Bxy^3}{6} + Exy + Fy + Hx + K. \tag{6.31'}$$

The terms of the first degree $Hx+Fy+K$ may be omitted because they do not apparently affect the stresses. One can easily see that function (6.31′) satisfies equation (IX) (Sec. 39) for any values of the coefficients, i.e., this function is its solution.

The stresses will be expressed as

$$\left.\begin{aligned} X_x &= \frac{\partial^2\varphi}{\partial y^2} = Ay + Bxy, \\ Y_y &= \frac{\partial^2\varphi}{\partial x^2} = 0, \\ X_y &= -\frac{\partial^2\varphi}{\partial x\,\partial y} = -\frac{B}{2}y^2 - E. \end{aligned}\right\} \tag{6.29a}$$

This shows that stresses of the form (6.29) or of the more general form (6.30) satisfy the equations of the theory of elasticity. We shall attempt now to select the indefinite coefficients A, B and E so as to satisfy the conditions on the surface (Fig. 51).

1. On the upper and lower faces

for $y = \frac{h}{2}$ $\quad Y_y = 0, \quad X_y = 0;$

for $y = -\frac{h}{2}$ $\quad Y_{-y} = -Y_y = 0, \quad X_{-y} = -X_y = 0.$

2. On the right-hand end section

$$\text{for } x = l \quad X_x = 0.$$

Since the law of distribution of shearing stresses over the section $x = l$ is in no way stipulated in the conditions of the problem, there remains only one condition

for $x = l$

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} Y_x\, dy = Q,$$

requiring that the forces at the end cross section reduce to the given load Q.

The reader is recommended to carry out the necessary computations as an exercise and to confirm for himself that the stresses will be represented by expressions (6.29).

Thus, in this problem the expressions for stresses (6.29) obtained on the basis of the hypothesis of plane sections are supported from the viewpoint of the theory of elasticity as well, provided the load Q is distributed over the end section in accordance with the law indicated by the last of formulas (6.29). When the load Q is applied in a different manner, the expressions for stresses will also be different, but a considerable numerical difference will be only near the loaded right-hand end of the plate, on the basis of Saint-Venant's principle (Sec. 31, Fig. 39).

In this problem it is of interest to investigate strains and displacements and to compare the result with that given by the hypothesis of plane sections in strength of materials. Since we shall now make use of Hooke's law, let us first decide which of the two cases (plane strain or plane stress) we shall dwell on; to obtain full agreement with an analogous problem in strength of materials let us treat a state of plane stress and use Hooke's law in the form (V′p) (Sec. 39). Then from (IIIp), (V′p) and (6.29) we obtain the following system of equations for determining displacements:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{Q}{EJ}(ly - xy), \\ \frac{\partial v}{\partial y} &= \frac{\sigma Q}{EJ}(ly - xy); \end{aligned}\right\} \tag{6.34}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{(1+\sigma)\, Q}{EJ}\left(\frac{h^2}{4} - y^2\right). \tag{6.35}$$

Integrating equations (6.34), we get

$$\left.\begin{aligned} u &= -\frac{Q}{EJ}\left(lyx - \frac{x^2y}{2}\right) + \frac{Q}{EJ} f_1(y), \\ v &= \frac{\sigma Q}{EJ}\left(\frac{ly^2}{2} - \frac{xy^2}{2}\right) + \frac{Q}{EJ} f_2(x), \end{aligned}\right\} \tag{6.36}$$

where f_1 and f_2 are arbitrary functions; for convenience in carrying out further computations, we have furnished them with the coefficient $\frac{Q}{EJ}$. To determine the form of these functions we substitute expressions (6.36) in equation (6.35); first we find

$$\left.\begin{aligned} \frac{\partial v}{\partial x} &= -\frac{\sigma Q}{EJ}\frac{y^2}{2} + \frac{Q}{EJ} f_2'(x), \\ \frac{\partial u}{\partial y} &= -\frac{Q}{EJ}\left(lx - \frac{x^2}{2}\right) + \frac{Q}{EJ} f_1'(y). \end{aligned}\right\} \tag{6.37}$$

Substituting this in equation (6.35) and cancelling out $\frac{Q}{EJ}$, we obtain

$$-\frac{\sigma y^2}{2} + f_2'(x) - \left(lx - \frac{x^2}{2}\right) + f_1'(y) = (1+\sigma)\frac{h^2}{4} - (1+\sigma)\, y^2$$

or after transposition of the terms

$$\left[f_2'(x) - \left(lx - \frac{x^2}{2}\right)\right] + \left[f_1'(y) - \frac{\sigma y^2}{2} + (1+\sigma)\, y^2\right] = (1+\sigma)\frac{h^2}{4}. \tag{6.38}$$

The bracketed functions depend: the first one on x alone, the second one on y alone; but since x and y are arbitrary and independent of each other, equality (6.38) can exist only if the bracketed expressions are equal to constant numbers m and n, so that

$$m + n = (1+\sigma)\frac{h^2}{4}. \tag{6.39}$$

On this basis we obtain from equation (6.38)

$$f_2'(x) = lx - \frac{x^2}{2} + m,$$

$$f'(y) = \frac{\sigma y^2}{2} - (1+\sigma)\, y^2 + n = -\left(1 + \frac{\sigma}{2}\right) y^2 + n.$$

Integrating these equations, we have

$$\left.\begin{aligned} f_2(x) &= \frac{lx^2}{2} - \frac{x^3}{6} + mx + \alpha, \\ f_1(y) &= -\frac{2+\sigma}{6}\, y^3 + ny + \beta. \end{aligned}\right\} \tag{6.40}$$

We introduce this in equations (6.36):

$$\left.\begin{aligned} u &= \frac{Q}{EJ}\left[-lxy + \frac{x^2y}{2} - \frac{2+\sigma}{6}y^3 + ny + \beta\right], \\ v &= \frac{Q}{EJ}\left[\frac{\sigma l y^2}{2} - \frac{\sigma x y^2}{2} + \frac{lx^2}{2} - \frac{x^3}{6} + mx + \alpha\right]. \end{aligned}\right\} \quad (6.41)$$

To determine the arbitrary constants m, n, α and β let us fix the left-hand end; in strength of materials all considerations are referred to the axis of a rod; therefore, here too we first fix the initial point of the axis, i.e., we specify the conditions

$$\text{for } x = y = 0 \quad u = 0; \quad v = 0.$$

Now from equations (6.41) we immediately have $\alpha = \beta = 0$.

It is necessary next to fix (Fig. 51) the supporting section (against rotation about the point O); for this purpose the condition that the initial tangent be horizontal is most frequently specified:

$$\text{for } \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \frac{\partial v}{\partial x} = 0\,^{1};$$

in this case we find from equations (6.41)

$$m = 0,$$

and from equation (6.39)

$$n = \frac{(1+\sigma)h^2}{4}.$$

Equations (6.41) give

$$u = \frac{Q}{EJ}\left[-\left(l - \frac{x}{2}\right)xy - \frac{(2+\sigma)y^3}{6} + \frac{(1+\sigma)h^2y}{4}\right], \quad (6.42)$$

$$v = \frac{Q}{EJ}\left[\frac{\sigma(l-x)y^2}{2} + \frac{lx^2}{2} - \frac{x^3}{6}\right]. \quad (6.43)$$

From equation (6.43), setting $y=0$, we find the equation of the deflected axis of the rod:

$$v = \frac{Q}{EJ}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right), \quad (6.44)$$

which coincides with the solution obtained in strength of materials. Now investigate the deformation of plane cross sections; let the equation of such a section before deformation be

$$x = x_0,$$

and let its equation after deformation be

$$x = x_0 + u_0,$$

[1] Cf. this with formula (2.3) in Sec. 10.

or

$$x = x_0 + \frac{Q}{EJ}\left[-\left(l - \frac{x_0}{2}\right)x_0 y - \frac{(2+\sigma)}{6}y^3 + \frac{(1+\sigma)h^2}{4}y\right]. \quad (6.45)$$

Consequently, the section does not remain plane but distorts along a parabola of the third order (6.45). In the above fixing conditions at the left-hand end (Fig. 52) the left-hand end section ($x_0=0$) also distorts along the curve

$$x = \frac{Q}{EJ}\left[\frac{(1+\sigma)h^2}{4}y - \frac{(2+\sigma)}{6}y^3\right].$$

If we take a linear element dy on the axis of the rod at the point of fixing (at $x=0$, $y=0$), its angle of rotation [formula (2.4) in Sec. 10] will be

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=0\\y=0}} = \frac{(1+\sigma)Qh^2}{4EJ} = \frac{3Q}{2Gh} > 0.$$

Consequently, the element rotates (Sec. 10) in the direction from the positive axis Oy to the positive axis Ox; thus, plane cross

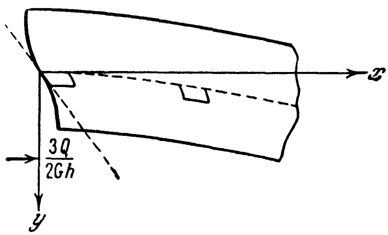

Fig. 52

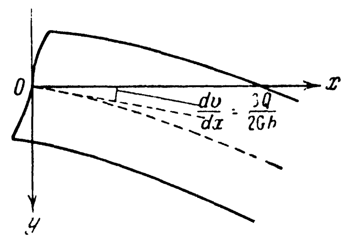

Fig. 53

sections distort after deformation and do not remain normal to the axis[1]. This shows that in our problem the formula of the normal stress

$$X_x = \frac{M}{J}y,$$

derived from the hypothesis of plane sections, holds valid when sections distort.

[1] One can easily find the general formula for the inclination of the section with respect to the axis at all its points; this inclination is obviously equal (Fig. 52) to the shear

$$(e_{xy})_{y=0} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_{y=0}.$$

It can most easily be calculated according to (6.35).

In Sec. 31 we pointed to the schematisation encountered sometimes in the formulation of problems with regard to load; it is useful to note here the same schematisation in the formulation of the "fixing conditions at the support". Indeed, fixing can be realised in an infinite number of manners: one may, for instance, require that all points of the section remain immovable or one may fix only several points of the section in the same sense; finally, one may add to these conditions the requirement that some linear elements should not rotate due to deformation. It is obvious that the stress distribution near the fixed end of the cantilever will depend in such conditions on the mode of fixing. The solution given above, however, has fully predetermined the stress distribution in the whole cantilever. To realise the fixing there remains at our disposal only the *rigid body displacement of the cantilever*, which does not affect the stress distribution; therefore, in this problem the concept of fixing is of a kinematic character only and the modes of fixing are limited.

In the preceding discussion we have fixed the left-hand end requiring that the element of the axis dx at the supporting end must be horizontal

$$\left(\frac{\partial v}{\partial x}=0 \quad \text{for} \quad x=y=0\right).$$

Let us now realise the same fixing in a different manner (Fig. 53) specifying the condition

$$\left(\frac{\partial u}{\partial y}\right)_{x=y=0}=0,$$

which requires [formula (2.4) in Sec. 10] that the element dy of the supporting section must remain vertical.

Then the first of equations (6.41) gives $n=0$ while from equation (6.39)

$$m=\frac{(1+\sigma)h^2}{4};$$

from equations (6.41) we obtain the following equations in place of (6.42) and (6.43):

$$u=\frac{Q}{EJ}\left[-\left(l-\frac{x}{2}\right)xy-\frac{(2+\sigma)y^3}{6}\right],$$

$$v=\frac{Q}{EJ}\left[\frac{\sigma(l-x)y^2}{2}+\frac{lx^2}{2}-\frac{x^3}{6}+\frac{(1+\sigma)h^2x}{4}\right].$$

Thus, all displacements have altered. The equation of the deflected axis will be [cf. formula (6.44)]:

$$v=\frac{Q}{EJ}\left[\frac{lx^2}{2}-\frac{x^3}{6}+\frac{(1+\sigma)h^2x}{4}\right]. \tag{6.46}$$

The deflection of the right-hand end in the first case [according to formula (6.44)] is

$$f = \frac{Ql^3}{3EJ};$$

for the second mode of fixing, however, it is given (Fig. 53) as

$$f = \frac{Ql^3}{3EJ} + \frac{(1+\sigma)\,Qh^2l}{4EJ} = \frac{Ql^3}{3EJ} + \frac{3Ql}{2Gh}. \tag{6.47}$$

The last term $\frac{3Ql}{2Gh}$ accounts for the effect of shear deformations (in other words, the effect of shearing force) on the deflection.

From the preceding discussion it is clear that the concept of the deflection of right-hand end as such is indefinite in this problem; the deflection may be different depending on the manner in which the left-hand end is "fixed".

In addition to the above two modes of fixing one can imagine an infinite number of other modes; corresponding to each of them, there will be certain specific displacements and certain deflection. Saint-Venant's principle states that the stresses vary substantially only in the neighbourhood of the fixed section; the displacements will be different, however, throughout the length of the rod.

42. BEAM ON TWO SUPPORTS

Consider the bending of a beam supported at the ends and subjected to a uniformly distributed load q. The supporting reactions are assumed to be the shearing forces distributed over the end sections. If the co-ordinate axes are located as in Fig. 54 the elementary solution of the problem leads to the following stresses:

$$X_x = \frac{\frac{q}{2}\left(\frac{l^2}{4} - x^2\right)}{J}\,y;$$

$$X_y = Y_x = -\frac{qx\left(\frac{h^2}{4} - y^2\right)}{2J}.$$

Let us write these formulas in a more general form:

$$\left.\begin{aligned} X_x &= Ay + Bx^2y, \\ X_y &= Y_x = Cx + Dxy^2, \end{aligned}\right\} \tag{6.48}$$

and check whether they satisfy all the equations of the plane problem. The stresses Y_y are usually neglected; this circumstance permits us to state a priori that the system of stresses (6.48) in

conjunction with the assumption $Y_y=0$ apparently will not satisfy the equations of the theory of elasticity as on the upper surface $y=-\frac{h}{2}$ we have the equality

$$Y_{-y}=-Y_y=q\neq 0.$$

We shall, therefore, attempt to satisfy the equations of the theory of elasticity prescribing stresses (6.48) but rejecting the condition $Y_y=0$.

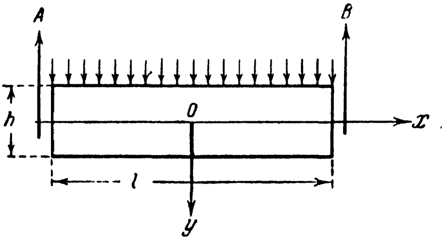

Fig. 54

First we select the general form of the stress function for stresses (6.48); according to equations (VIII) (Sec. 39) we have

$$\left.\begin{aligned}\frac{\partial^2\varphi}{\partial y^2}&=Ay+Bx^2y,\\ \frac{\partial^2\varphi}{\partial x\,\partial y}&=-Cx-Dxy^2.\end{aligned}\right\} \tag{6.49}$$

By integrating the first of these equations we get

$$\frac{\partial\varphi}{\partial y}=\frac{A}{2}y^2+\frac{B}{2}x^2y^2+f_1(x), \tag{6.50}$$

$$\varphi=\frac{Ay^3}{6}+\frac{B}{6}x^2y^3+f_1(x)\,y+f_2(x), \tag{6.51}$$

where f_1 and f_2 are arbitrary functions of x. Differentiating equation (6.50) with respect to x and substituting the resulting value in the second equation of (6.49), we have

$$\frac{\partial^2\varphi}{\partial x\,\partial y}=Bxy^2+f_1'(x)=-Cx-Dxy^2;$$

hence we obtain

$$D=-B;\quad f_1(x)=-\frac{C}{2}x^2+E.$$

Introducing this value in equality (6.51), we have

$$\varphi=\frac{A}{6}y^3+\frac{B}{6}x^2y^3-\frac{C}{2}x^2y+Ey+f_2(x).$$

Substituting this expression in equation (IX) (Sec. 39), we see that it is not satisfied [1]; thus, φ cannot be taken as a stress function; we add, therefore, an arbitrary function $\psi(x, y)$ [2] to obtain

$$\varphi=\frac{A}{6}y^3+\frac{B}{6}x^2y^3-\frac{C}{2}x^2y+f_2(x)+\psi(x, y). \tag{6.52}$$

The function $\psi(x, y)$ is selected so as to satisfy the compatibility condition (IX). From equality (6.52) we have

$$\frac{\partial^4\psi}{\partial x^4}=\frac{\partial^4\varphi}{\partial x^4};\quad 2\frac{\partial^4\varphi}{\partial x^2\,\partial y^2}=4By+\frac{\partial^4\psi}{\partial x^2\,\partial y^2};\quad \frac{\partial^4\varphi}{\partial y^4}=\frac{\partial^4\psi}{\partial y^4},$$

and equation (IX) becomes (assuming $f_2^{\mathrm{IV}}(x)\equiv 0$)

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\,\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=-4By. \tag{6.53}$$

Looking at this equation we can easily see that its simplest solution will be represented by an integral function of the fifth degree:

$$\psi(x, y)=\frac{F}{24}x^4y+\frac{H}{120}y^5+\frac{K}{12}x^2y^3.$$

Introducing this in equation (6.53), we have

$$Fy+2Ky+Hy=-4By,$$

i.e.,

$$F+2K+H=-4B. \tag{6.54}$$

The term $\frac{K}{12}x^2y^3$ in the function ψ may be omitted since equality (6.52) already contains the term of a similar form $\frac{B}{6}x^2y^3$; then from equation (6.54) we have

$$H=-4B-F.$$

After that the stress function (6.52) takes the final form

$$\varphi=\frac{Ay^3}{6}+\frac{Bx^2y^3}{6}-\frac{C}{2}x^2y+f_2(x)+\frac{F}{24}x^4y-\frac{(4B+F)}{120}y^5 \tag{6.55}$$

which satisfies condition (IX). Hence we obtain stresses

$$\left.\begin{aligned} X_x&=Ay+Bx^2y-\frac{4B+F}{6}y^3,\\ Y_y&=\frac{B}{3}y^3-Cy+f_2''(x)+\frac{F}{2}x^2y,\\ X_y&=Y_x=-Bxy^2+Cx-\frac{F}{6}x^3. \end{aligned}\right\} \tag{6.56}$$

---

[1] It will become $4By=0$.
[2] The term Ey, which does not affect the stresses, is omitted.

To complete the solution it remains to satisfy the conditions on the surface; first consider the conditions on the upper and lower faces.

1. On the upper face $y = -\frac{h}{2}$

$$Y_{-y} = -Y_y = q; \quad X_y = -X_{-y} = 0. \tag{6.57}$$

2. On the lower face $y = +\frac{h}{2}$

$$Y_y = 0; \quad X_y = 0. \tag{6.58}$$

The first of conditions (6.58) gives

$$\frac{Bh^3}{24} - \frac{Ch}{2} + f_2''(x) + \frac{Fh}{4}x^2 = 0.$$

Hence we obtain

$$F = 0; \quad f_2''(x) = L,$$

and stresses (6.56) will be

$$\left.\begin{aligned} X_x &= Ay + Bx^2y - \frac{2}{3}By^3, \\ Y_y &= \frac{B}{3}y^3 - Cy + L, \\ X_y &= Y_x = -Bxy^2 + Cx. \end{aligned}\right\} \tag{6.59}$$

From conditions (6.57) and (6.58) for normal stresses we get

$$-\frac{Bh^3}{24} + \frac{Ch}{2} + L = -q; \quad \frac{Bh^3}{24} - \frac{Ch}{2} + L = 0.$$

Adding up these equations, we find

$$L = -\frac{q}{2}; \tag{6.60}$$

subtracting them, we obtain

$$\frac{Bh^3}{12} - Ch = q. \tag{6.61}$$

Conditions (6.57) and (6.58) for shearing stresses yield

$$-\frac{Bh^2}{4} + C = 0; \tag{6.62}$$

solving equations (6.61) and (6.62) simultaneously, we find

$$B = -\frac{6q}{h^3}; \quad C = -\frac{3}{2}\frac{q}{h}. \tag{6.63}$$

Substituting the values of L, B and C from (6.60) and (6.63) in formulas (6.59), we obtain

$$\left.\begin{aligned} X_x &= Ay - \frac{6q}{h^3}x^2y + 4\frac{q}{h^3}y^3, \\ Y_y &= -\frac{6q}{h^3}\left(\frac{y^3}{3} - \frac{h^2}{4}y + \frac{h^3}{12}\right), \\ X_y &= -\frac{6q}{h^3}\left(\frac{h^2}{4} - y^2\right)x. \end{aligned}\right\} \tag{6.64}$$

We proceed to the conditions on the end sections of the rod:

$$\left.\begin{aligned} &\text{for } x = \pm\frac{l}{2} \quad X_x = 0, \\ &\int_{-\frac{h}{2}}^{+\frac{h}{2}} Y_x\,dy = -\frac{ql}{2}. \end{aligned}\right\} \tag{6.65}$$

The last condition requires that the shearing stresses on the end section reduce to the supporting reaction $\left(-\frac{ql}{2}\right)$; one can easily see that it is already satisfied, since the last of equations (6.64) coincides exactly with the prescribed expression of the shearing stress [formulas (6.48)]. At the same time we note that the first of conditions (6.65) cannot be satisfied; indeed, for $x = \pm\frac{l}{2}$

$$X_x = \left[\left(A - \frac{6ql^2}{4h^3}\right)y + 4\frac{q}{h^3}y^3\right] \tag{6.66}$$

and, consequently, X_x does not vanish; but since the right-hand member of equality (6.66) is an odd function of y, it follows that

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} X_x\,dy = 0, \tag{6.67}$$

i.e., the normal stresses over the end section reduce to a couple with the moment

$$M_0 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} X_x y\,dy = A\frac{h^3}{12} - \frac{ql^2}{8} + \frac{qh^2}{20}. \tag{6.68}$$

Thus, solution (6.64) corresponds to the bending of a plate under the action of a uniformly distributed load q and moments $\pm M_0$ at the supports (Fig. 55).

By a suitable adjustment of the coefficient A we can assign any value to the supporting moments (6.68). Expressing the coefficient

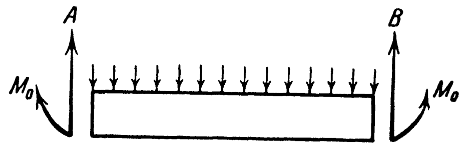

Fig. 55

A through the supporting moment M_0 according to equation (6.68), we have

$$A = \frac{12}{h^3}\left(M_0 + \frac{ql^2}{8} - \frac{qh^2}{20}\right)$$

and the expressions of stresses (6.64) will be written finally as

$$\left.\begin{aligned} X_x &= \frac{12}{h^3}\left(M_0 + \frac{ql^2}{8} - \frac{qh^2}{20}\right)y - \frac{6q}{h^3}x^2y + \frac{4q}{h^3}y^3, \\ Y_y &= -\frac{6q}{h^3}\left(\frac{y^3}{3} - \frac{h^2}{4}y + \frac{h^3}{12}\right), \\ X_y &= -\frac{6q}{h^3}\left(\frac{h^2}{4} - y^2\right)x. \end{aligned}\right\} \tag{6.69}$$

If we set here

$$M_0 = 0, \tag{6.70}$$

the supporting moments will be eliminated and we shall return to the bending of a plate under a continuous load alone, but in the presence of normal stresses (6.66) on the supporting sections; on the basis of equations (6.67) and (6.70), however, these stresses reduce to a balanced system of forces and, therefore, their influence, by virtue of Saint-Venant's principle (Sec. 31), will be appreciable only near the ends of the beam. For $M_0 = 0$ the first of equations (6.69) can be written down as

$$X_x = \frac{12q}{h^3}\left[\frac{1}{2}\left(\frac{l^2}{4} - x^2\right)y - \frac{h^2}{20}y + \frac{y^3}{3}\right]. \tag{6.71}$$

Comparing this with the first of equations (6.48), we note that the difference in the expressions of the stress

$$\frac{12q}{h^3}\left(\frac{h^2}{20}y - \frac{y^3}{3}\right) = \frac{q}{J}\left(\frac{h^2}{20}y - \frac{y^3}{3}\right)$$

will have only a little effect on the magnitude of stresses in the middle part of the beam (for small x) when the height of the section h is small compared with the span l.

Summarising the foregoing conclusions, we can say that:

1. The expression of shearing stresses given by the elementary solution is confirmed by the present more exact solution (6.69).

2. For the normal stress X_x, the elementary solution gives values close to reality far from the ends of a slab or a plate (in the case when the height h is small compared with the span), where these stresses play a dominant part.

3. The second of equations (6.69) gives the expression of the normal stress Y_y which is neglected in the elementary solution. The diagram of the distribution of these stresses across the depth of the section (cubic parabola) is shown in Fig. 56. By using equations (6.69), one can obviously solve the problem of a beam with the ends built into walls; for that the magnitude of the supporting moments M_0 must be chosen in such a way as to satisfy the conditions of the form $\frac{\partial v}{\partial x}=0$ or $\frac{\partial u}{\partial y}=0$ (cf. Sec. 41) at the ends of the axis of the beam.

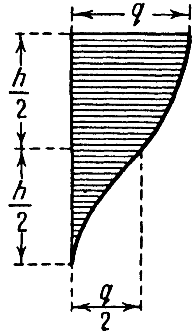

Fig. 56

First it is necessary, of course, to find the displacements of the beam, as done in Sec. 41.

43. TRIANGULAR AND RECTANGULAR RETAINING WALLS (M. LÉVY'S SOLUTIONS)

Consider a problem of a dam wall or a retaining wall subjected to the pressure of water or a loose material according to the hydrostatic law in proportion to the depth of a given area (Fig. 57). The conditions on the surface will obviously be expressed as:

1. On the face Oy for $x=0$

$$\left.\begin{aligned} Y_{-x} &= -Y_x = 0, \\ X_{-x} &= -X_x = \gamma y. \end{aligned}\right\} \tag{6.72}$$

2. On the face OA for $x=y \tan \beta$

$$X_v=0; \quad Y_v=0. \tag{6.73}$$

Since the number of conditions on the surface is four, for the solution of the problem it is sufficient to take a stress function with four indefinite coefficients. Let us make use of the function

of the third degree (6.26) retaining the first four terms in it, i.e., setting

$$a = b = c = 0.$$

According to equations (6.27) (Sec. 40) the stresses will be linear functions and will be expressed as

$$\left.\begin{aligned} X_x &= fx + ky; \quad Y_y = dx + ey; \\ Y_x &= X_y = -ex - fy - \rho g x. \end{aligned}\right\} \tag{6.74}$$

Here ρg is the weight per unit volume of the wall; for brevity designate it by p, i.e.,

$$\rho g = p.$$

Applying conditions (6.72), we have: $fy=0$, $ky= -\gamma y$. Hence $f=0$, $k= -\gamma$ and stresses (6.74) will be written as

$$X_x = -\gamma y; \quad Y_y = dx + ey; \quad Y_x = X_y = -(e+p)x. \tag{6.74'}$$

We write out conditions (6.73) according to equations (IIp)

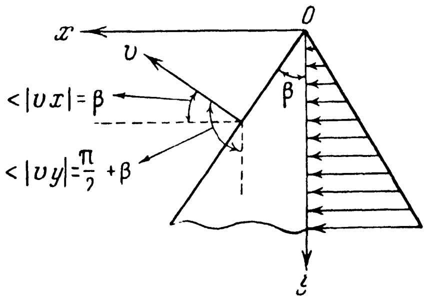

Fig. 57

(Sec. 38) and introduce in them the values of the cosines of the angles between the outward normal v and the axes (Fig. 57):

$$X_x \cos\beta - X_y \sin\beta = 0; \quad Y_x \cos\beta - Y_y \sin\beta = 0.$$

We introduce here the values of stresses (6.74′) setting, in accordance with conditions (6.73), $x=y\tan\beta$

$$-\gamma y \cos\beta + (e+p)\, y \tan\beta \sin\beta = 0,$$

$$-(e+p)\, y \tan\beta \cos\beta - (dy \tan\beta + ey) \sin\beta = 0.$$

Cancelling y out of both equations and dividing by $\cos\beta$, we obtain

$$(e+p)\tan^2\beta = \gamma; \quad (d\tan\beta + e)\tan\beta = -(e+p)\tan\beta.$$

Hence we easily find

$$e=\frac{\gamma}{\tan^2\beta}-p; \quad d=\frac{p}{\tan\beta}-\frac{2\gamma}{\tan^3\beta},$$

and stresses (6.74) assume the final form:

$$\left.\begin{aligned} X_x &= -\gamma y, \\ Y_y &= \left(\frac{p}{\tan\beta}-\frac{2\gamma}{\tan^3\beta}\right)x+\left(\frac{\gamma}{\tan^2\beta}-p\right)y, \\ Y_x &= X_y = -\frac{\gamma}{\tan^2\beta}x. \end{aligned}\right\} \qquad (6.75)$$

The diagrams of stresses Y_y and X_y over the horizontal section $y=y_0$ are shown in Fig. 58. Comparing this with the result of the elementary calculation according to the formulas

$$Y_y=\frac{N}{F}+\frac{M}{J}\left(x-\frac{a}{2}\right), \quad X_y=\frac{QS}{Jb},$$

we find that the values of the normal stresses Y_y coincide in both solutions; the distribution of the shearing stresses X_y, however, appears to be essentially different.

Hitherto we have selected the stress function $\varphi(x, y)$ in the form of a polynomial of not higher than the fifth degree [cf. formula (6.55) in Sec. 42].

By increasing the degree of a polynomial, it is possible to find the solution of more complicated problems. For instance, prescribing φ as a polynomial of the sixth degree, one can obtain the solution given by M. Lévy for the case of a rectangular dam or retaining wall. This solution has the following form:

$$\left.\begin{aligned} X_x &= -\gamma y\left(1-\frac{x}{a}\right)^2\left(1+2\frac{x}{a}\right), \\ Y_y &= -py-\frac{\gamma y^3}{a^2}\left(2\frac{x}{a}-1\right)+\gamma y\left(4\frac{x^3}{a^3}-6\frac{x^2}{a^2}+\frac{12}{5}\frac{x}{a}-\frac{1}{5}\right), \\ Y_x &= X_y = -\frac{\gamma x}{a}\left(1-\frac{x}{a}\right)\left[3\frac{y^2}{a}-\frac{a}{5}+x\left(1-\frac{x}{a}\right)\right]. \end{aligned}\right\} \qquad (6.76)$$

The reader can easily prove that these stresses satisfy the following conditions on the surface (Fig. 59):

1. For $x=0$

$$X_{-x}=\gamma y; \quad Y_{-x}=0.$$

2. For $x=a$

$$X_x=0; \quad Y_x=0.$$

3. For $y=0$

$$Y_{-y}=0.$$

The last necessary condition

$$\text{for } y=0 \quad X_{-y}=0$$

is not satisfied completely and on the upper face of the wall there remain shearing stresses

$$X_{-y}=-X_y=\frac{\gamma x}{a}\left(1-\frac{x}{a}\right)\left[x\left(1-\frac{x}{a}\right)-\frac{a}{5}\right].$$

These stresses, however, reduce to a balanced system of forces since their resultant is equal to zero:

$$\int_0^a X_y\,dx=0;$$

they have therefore only local significance and the region of their influence, according to Saint-Venant's principle, is small, the

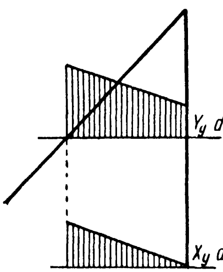

Fig. 58

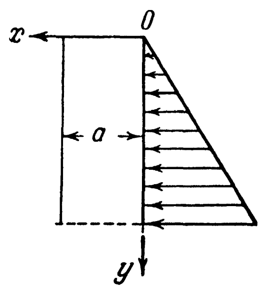

Fig. 59

more so as they act in the upper, less stressed part of the wall where the stress analysis is not usually required.

Solving this problem by the elementary method, we would obtain the following expression for the stresses Y_y:

$$Y_y=\frac{N}{F}+\frac{M}{J}\left(x-\frac{a}{2}\right)=$$

$$=-py-\frac{\frac{\gamma y^2}{2}\,\frac{y}{3}}{\left(\frac{a^3}{12}\right)}\,\frac{a}{2}\left(2\,\frac{x}{a}-1\right)=-py-\frac{\gamma y^3}{a^2}\left(2\,\frac{x}{a}-1\right),$$

which corresponds to the first two terms of the second of equations (6.76).

The reader is recommended to compare, in a similar manner, the expressions of the stress $X_y = Y_x$ in the elementary solution and according to equations (6.76).

44. BENDING OF A RECTANGULAR STRIP; FILON'S AND RIBIÈRE'S SOLUTIONS

The method of solution of the plane problem by means of algebraic polynomials, considered in the preceding sections, affords limited possibilities for practical application because it is very difficult to select a polynomial which gives the solution corresponding to a prescribed load which is more or less complicated. The method of trigonometric polynomials, proposed by C. Ribière and L.N.G. Filon for the case of bending a rectangular strip whose length is considerably greater than the height h (Fig. 60),

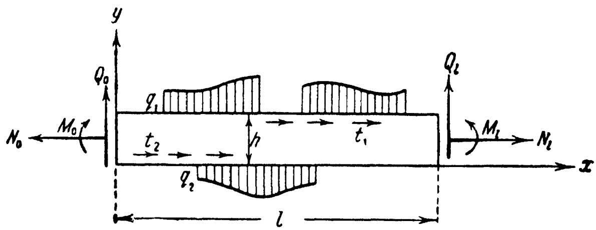

Fig. 60

was found to be far more effective. In such cases it is most important to satisfy as exactly as possible the conditions on the long sides of the strip where a prescribed load producing the bending of the strip acts; the conditions on the short sides (ends) can be satisfied only by prescribing the resultant factors N_0, Q_0, M_0, N_l, Q_l, M_l, which characterise the reactions of the supports or the loads at the ends of the strip (without regard to the law of stress distribution across the height h of the initial and terminal sections); when the ratio $\frac{h}{l}$ is small this is permissible from the point of view of Saint-Venant's principle (Sec. 31).

As indicated in Sec. 39, the solution of the plane problem by means of the stress function $\varphi(x, y)$ is reduced to the integration of equation (IX)

$$\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\,\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0 \qquad (6.77)$$

with the prescribed boundary conditions; these conditions will be dealt with later. Now we shall attempt to obtain as general a solution of equation (6.77) as possible. Such a solution can easily be found by the method of separation of variables used above in Sec. 28 with respect to the equation of vibrations (4.28); Ribière's and Filon's methods are particular cases.

Let us seek a particular solution of equation (6.77) in the form

$$\varphi(x, y) = XY, \tag{6.78}$$

where X is a function of x only, and Y is a function of y only. Substituting (6.78) in (6.77), we write it down as

$$X^{(IV)}Y + 2X''Y'' + XY^{(IV)} = 0. \tag{6.79}$$

We shall eliminate the function X from here; for that it is sufficient to require that $X^{(IV)}$ and X'' be proportional to X:

$$X^{(IV)} = \alpha^4 X; \quad X'' = -\lambda^2 X, \tag{6.80}$$

where α^4 and λ^2 are certain constants. Thus, the function X must satisfy simultaneously two differential equations (6.80). Let us examine in what conditions they will be consistent; for this purpose we differentiate the second equation twice and, comparing the result with the first equation, we obtain

$$X^{(IV)} = -\lambda^2 X'' = \alpha^4 X.$$

The first equation (6.80) can then be replaced by

$$-\lambda^2 X'' = \alpha^4 X,$$

and instead of (6.80) we consider the system

$$-\lambda^2 X'' = \alpha^4 X; \quad X'' = -\lambda^2 X.$$

It follows that $\frac{\alpha^4}{\lambda^2} = \lambda^2$, i.e., we may assume $\alpha = \lambda$ and the system (6.80) will be replaced by

$$X^{(IV)} = \lambda^4 X; \quad X'' = -\lambda^2 X. \tag{6.81}$$

Substituting this in (6.79), we eliminate X from it and obtain the equation of the fourth order for determining Y:

$$Y^{(IV)} - 2\lambda^2 Y'' + \lambda^4 Y = 0. \tag{6.82}$$

The function X will be found from the second equation of (6.81):

$$X'' + \lambda^2 X = 0. \tag{6.83}$$

Its general solution is well known:

$$X = K_1 \cos \lambda x + K_2 \sin \lambda x. \tag{6.84}$$

We now concentrate attention on equation (6.82). Seeking its particular solutions in the form

$$Y = e^{sy},$$

we obtain the characteristic equation

$$(s^2 - \lambda^2)^2 = 0, \tag{6.82a}$$

having two double roots

$$s = \pm \lambda.$$

Hence we obtain the general solution of equation (6.82) in the usual way in the form

$$Y = A \cosh \lambda y + B \sinh \lambda y + Cy \cosh \lambda y + Dy \sinh \lambda y, \tag{6.85}$$

where A, B, C, D are arbitrary constants.

Substituting the values (6.83) and (6.85) in (6.78), we find the following particular solution of the equation of the plane problem (6.77):

$$\varphi(x, y) = (K_1 \cos \lambda x + K_2 \sin \lambda x)(A \cosh \lambda y + B \sinh \lambda y + \\ + Cy \cosh \lambda y + Dy \sinh \lambda y). \quad (6.8\ )$$

This function is a solution of equation (6.77) for arbitrary values of the constants

$$K_1, \ K_2, \ A, \ B, \ C, \ D \text{ and } \lambda.$$

Consequently, it is possible to construct any number of solutions of the type (6.86); the sum of such solutions will also be a solution of equation (6.77) (since this equation is linear). Taking a sufficiently large number of terms of this sum, we shall have many arbitrary constants at our disposal; later we shall try to select them so as to satisfy the conditions prescribed at the boundaries of the strip as fully as possible.

First consider the conditions at the ends of the strip having length l (Fig. 60). Let us require that at these sections, i.e.,

$$\text{for } x=0 \quad \text{and} \quad x=l$$

no normal stresses occur; in other words, we require that

$$\left.\begin{array}{l}\text{for } \ x=0\\ \text{and } x=l\end{array}\right\} X_x=\frac{\partial^2\varphi}{\partial y^2}=0. \tag{6.87}$$

There will still remain shearing stresses at the ends, and the loads on the long sides of the strip will be balanced by the forces Q_0 and Q_l applied at the ends of the beam; these forces can be considered to be the reactions of the supports of a simple beam of length l.

Conditions (6.87) will be satisfied if we set in the particular solution (6.86)

$$K_1=0; \quad K_2=1; \quad \lambda=\frac{m\pi}{l},$$

where m is zero or a positive integer: $m=0, 1, 2, 3, \ldots, \infty$. Let us form the sum of particular solutions taking

$$m=1, 2, 3, \ldots, n$$[1]:

$$\varphi=\sum_{m=1}^{m=n}\sin\frac{m\pi x}{l}\left(A_m\cosh\frac{m\pi y}{l}+B_m\sinh\frac{m\pi y}{l}+\right.$$
$$\left.+C_m y\cosh\frac{m\pi y}{l}+D_m y\sinh\frac{m\pi y}{l}\right). \tag{6.88}$$

There appear here $4n$ arbitrary constants A_m, B_m, C_m, D_m which, for a sufficiently great number n, will enable us to satisfy the conditions on the long sides of the strip if loads are prescribed there. The solution in the form (6.88) was first proposed by L.N.G. Filon.

We proceed to the conditions on the long sides. Let an arbitrary normal load $q_1=q_1(x)$ and a tangential load $t_1=t_1(x)$ be given on the upper side $(y=h)$; let a normal load $q_2=q_2(x)$ and a tangential load $t_2=t_2(x)$ act on the lower side $(y=0)$. As regards the tangential loads, we assume that the sum of each of them separately is equal to zero, i.e.,

$$\int_0^l t_1(x)\,dx=0, \quad \int_0^l t_2(x)\,dx=0. \tag{6.89}$$

[1] The value $m=0$ is omitted for the time being and will be considered later.

Then we obtain the following boundary conditions for the stress function:

$$\left.\begin{aligned}&\text{for } y=h \quad Y_y=\frac{\partial^2\varphi}{\partial x^2}=q_1(x), \qquad X_y=-\frac{\partial^2\varphi}{\partial x\,\partial y}=t_1(x);\\ &\text{for } y=0 \quad -Y_y=-\frac{\partial^2\varphi}{\partial x^2}=q_2(x), \quad -X_y=+\frac{\partial^2\varphi}{\partial x\,\partial y}=t_2(x),\end{aligned}\right\} \quad (6.90)$$

or, substituting here the expression of $\varphi(x, y)$ from (6.88), we get

$$\left.\begin{aligned}&-\frac{\pi^2}{l^2}\sum_{m=1}^{n} m^2 \sin\frac{m\pi x}{l}\Big(A_m\cosh\frac{m\pi h}{l}+B_m\sinh\frac{m\pi h}{l}+\\ &\qquad\qquad + C_m h\cosh\frac{m\pi h}{l}+D_m h\sinh\frac{m\pi h}{l}\Big)=f_1(x);\\ &\frac{\pi}{l}\sum_{n=1}^{n} m\cos\frac{m\pi x}{l}\Big[\Big(A_m\frac{m\pi}{l}+D_m\Big)\sinh\frac{m\pi h}{l}+\\ &\qquad +\Big(B_m\frac{m\pi}{l}+C_m\Big)\cosh\frac{m\pi h}{l}+C_m h\sinh\frac{m\pi h}{l}+\\ &\qquad\qquad + D_m h\cosh\frac{m\pi h}{l}\Big]=t_1(x);\\ &\frac{\pi^2}{l^2}\sum_{m=1}^{n} m^2\sin\frac{m\pi x}{l}A_m=f_2(x);\\ &\frac{\pi}{l}\sum_{m=1}^{n} m\cos\frac{m\pi x}{l}\Big(B_m\frac{m\pi h}{l}+C_m\Big)=t_2(x).\end{aligned}\right\} \quad (6.90a)$$

Considering the second and fourth of these equations, we see the necessity of the restriction imposed on the loads t_1 and t_2 by conditions (6.89); indeed, the corresponding integrals of the left-hand members of these equalities taken between the limits $(0, l)$ vanish. In any case, the above equalities show that the functions appearing in the right-hand members must be represented approximately in the interval $(0, l)$ by Fourier series in $\sin\frac{m\pi x}{l}$ and $\cos\frac{m\pi x}{l}$ respectively. The greater the number of terms in the stress function (6.88), the more exact this representation. To determine the coefficients of the series we follow the general rule: we multiply both members of the equality by $\sin\frac{k\pi x}{l}$ or $\cos\frac{k\pi x}{l}$ respectively and integrate from 0 to l; in this way we obtain the following four equations for determining the coefficients A_k, B_k, C_k, D_k appearing in the kth term of the

stress function (6.88):

$$\left.\begin{aligned}
&A_k\cosh\frac{k\pi h}{l}+B_k\sinh\frac{k\pi h}{l}+C_k h\cosh\frac{k\pi h}{l}+D_k h\sinh\frac{k\pi h}{l}=\\
&\qquad\qquad=-\frac{2l}{\pi^2k^2}\int_0^l f_1(x)\sin\frac{k\pi x}{l}\,dx;\\
&A_k\frac{k\pi}{l}\sinh\frac{k\pi h}{l}+B_k\frac{k\pi}{l}\cosh\frac{k\pi h}{l}+\\
&\qquad+C_k\left(\cosh\frac{k\pi h}{l}+h\sinh\frac{k\pi h}{l}\right)+\\
&\quad+D_k\left(\sinh\frac{k\pi h}{l}+h\cosh\frac{k\pi h}{l}\right)=\frac{2}{\pi k}\int_0^l t_1(x)\cos\frac{k\pi x}{l}\,dx;\\
&\qquad A_k=\frac{2l}{\pi^2k^2}\int_0^l f_2(x)\sin\frac{k\pi x}{l}\,dx;\\
&\qquad B_k\frac{k\pi h}{l}+C_k=\frac{2}{\pi k}\int_0^l t_2(x)\cos\frac{k\pi x}{l}\,dx.
\end{aligned}\right\}\quad(6.90\text{b})$$

Setting here $k=1, 2, 3, \ldots, n$, we obtain equations for determining all the coefficients of the stress function (6.88). With

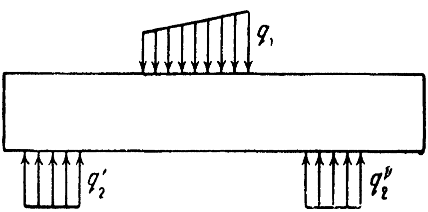

Fig. 61

the stress function we can find stresses at any point of the strip according to the formulas

$$X_x=\frac{\partial^2\varphi}{\partial y^2};\quad Y_y=\frac{\partial^2\varphi}{\partial x^2};\quad X_y=-\frac{\partial^2\varphi}{\partial x\,\partial y}.$$

The foregoing method of investigating the bending of a strip proved to be very convenient since, when using it, we can introduce sufficiently arbitrary loads on the long sides. In the courses on the theory of elasticity by S. P. Timoshenko and P. F. Papkovitch one can find many examples of application of this method to problems of practical importance.

If the tangential loads t_1 and t_2 are absent, we have the problem of the transverse bending of the strip by the usual transverse loads q_1 and q_2. Consider a particular case when the load $q_1=f_1(x)$ is prescribed arbitrarily, while the load $q_2=f_2(x)$ is distributed over two portions (Fig. 61); its intensities q_2' and q_2'' are chosen so that all these loads are mutually balanced. Though there still will remain tangential tractions at the ends of the beam, their

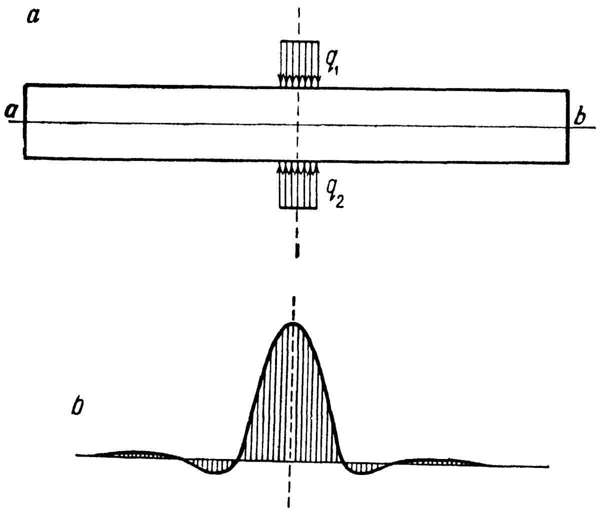

Fig. 62

resultants Q_0 and Q_l (Fig. 60) will vanish; the influence of these balanced tangential tractions Y_x, according to Saint-Venant's principle, will be appreciable only near the ends of the beam. In this way we shall obtain the solution of the problem of a beam resting on two supports, provided the reactions of the supports q_2' and q_2'' are distributed according to a preassigned law. A number of problems of this kind were solved by G. N. Maslov for simple and continuous beams.

Another interesting particular case will result if we take the loads q_1 and q_2 (Fig. 62) equal but opposite in sense (t_1 and t_2 are absent as before). The forces Q_0 and Q_l will be equal to zero and we shall obtain the compression of the strip in the lateral direction. If the loads q_1 and q_2 are distributed over a small portion (Fig. 62a), then by contracting it we shall obtain in the limit a scheme close to Fig. 38 which was used to illustrate Saint-Venant's principle. In the present problem this principle is con-

firmed by the shape of the diagram of stresses Y_y over the middle section ab, shown in Fig. 62b for the limiting case of compression of a strip by concentrated loads. The solution of this problem was given by Filon [1].

Ribière applied to some problems a solution analogous to (6.88) and obtainable in the same way from (6.86) if we set $K_2=0$, $\lambda=\frac{m\pi}{l}$; thus we have

$$\varphi=\sum_{m=1}^{m=n}\cos\frac{m\pi x}{l}\left(A_m\cosh\frac{m\pi y}{l}+B_m\sinh\frac{m\pi y}{l}+\right.$$
$$\left.+C_m y\cosh\frac{m\pi y}{l}+D_m y\sinh\frac{m\pi y}{l}\right). \quad (6.91)$$

A series of applications of this solution are given in the course on the theory of elasticity by P. F. Papkovitch.

In order to lift the restriction imposed on the tangential loads by conditions (6.89) in Filon's solution and to broaden the limits of its applicability in general, it is necessary to add to the solution (6.88) one more particular solution, resulting from the adopted method of separation of variables but so far omitted. Indeed, in addition to the values $m=1, 2, 3, \ldots$ used in (6.88) to construct the function φ, let us set $m=0$, i.e., $\lambda=\frac{m\pi}{l}=0$; then equations (6.83) and (6.82) become

$$X''=0,$$
$$Y^{(\mathrm{IV})}=0;$$

their general solutions are

$$X=\alpha x+\beta,$$
$$Y=\frac{A}{6}y^3+\frac{B}{2}y^2+Cy+D,$$

and according to (6.78) we obtain[2] the corresponding stress function

$$\varphi_0=(\alpha x+\beta)\left(\frac{A}{6}y^3+\frac{B}{2}y^2+Cy\right). \quad (6.92)$$

Thus, the solution (6.88) is complemented according to the above scheme by the solution in the form of a polynomial of the fourth degree (6.92); the components of the state of stress

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[1] E. G Coker, L. N. G. Filon, *A Treatise on Photoelasticity*, Cambridge, Univ. Press, 1931; S. P. Timoshenko, *Theory of Elasticity*, New York, McGraw-Hill Book Co., 1934.

[2] Henceforth we assume $D=0$.

will now be

$$\left.\begin{aligned} X_x &= \frac{\partial^2\varphi_0}{\partial y^2} = (\alpha x + \beta)(Ay + B), \\ Y_y &= \frac{\partial^2\varphi_0}{\partial x^2} = 0, \\ X_y &= -\frac{\partial^2\varphi_0}{\partial x\,\partial y} = -\alpha\left(\frac{A}{2}y^2 + By + C\right). \end{aligned}\right\} \tag{6.93}$$

We obtain the following stresses on the long sides of the strip

$$\left.\begin{aligned} &\text{for } y=0 \quad Y_y = 0,\ X_y = -\alpha C; \\ &\text{for } y=h \quad Y_y = 0,\ X_y = -\alpha\left(\frac{Ah^2}{2} + Bh + C\right). \end{aligned}\right\} \tag{6.94}$$

Consequently, by a suitable adjustment of the arbitrary constants α, β, A, B, C we can:

1. Obtain constant tangential loads X_y of arbitrary intensity according to (6.94) and thereby get rid of the restriction (6.89) on the long sides ($y=0$, $y=h$).

2. Obtain normal stresses at the ends of the strip:

$$\left.\begin{aligned} &\text{for } x=0 \quad X_x = \beta(Ay + B); \\ &\text{for } x=l \quad X_x = (\alpha l + \beta)(Ay + B), \end{aligned}\right\} \tag{6.95}$$

distributed according to a linear law; this permits us to add two more factors of arbitrary intensity to solution (6.88): compression-tension of the strip along its axis and bending moments at the ends.

*45. ONE MODIFICATION OF FILON'S METHOD[1]

The modification presented here may be found useful in cases where a given strip is to be designed under various loads and also in cases where it is necessary to determine the displacements of its upper or lower boundaries; the latter is required in the solution of *contact problems*; for instance, in the case of a strip resting on an absolutely rigid foundation and loaded over the upper face; sometimes it is required to calculate the displacements v (deflections) of the lower and upper faces of a strip supported at the ends.

To solve the problem stated in Sec. 44 it was necessary to represent the stress function $\varphi(x, y)$ in the form (6.88); the first

[1] See the author's paper *On the Bending of a Strip*, Vestn. Voen.-inzh. akad. im. V. V. Kuibysheva, 1936, No. 20.

factor under the summation sign $\sin\frac{m\pi x}{l}$ is a particular solution of equation (6.83); we shall leave it unchanged; the second factor (in parentheses) represents the general solution of equation (6.82) for $\lambda=\frac{m\pi}{l}$. It is composed of four particular solutions

$$\cosh\frac{m\pi y}{l};\quad \sinh\frac{m\pi y}{l};\quad y\cosh\frac{m\pi y}{l};\quad y\sinh\frac{m\pi y}{l} \tag{6.96}$$

obtained by the use of the characteristic equation (6.82a); however, instead of these we can take some other four particular, linearly independent solutions representing linear functions of solutions (6.96); let us write them down in the form

$$\left.\begin{aligned}
\psi_{1m}(z) &= a_{1m}\cosh z + b_{1m}\sinh z + c_{1m}z\cosh z + d_{1m}z\sinh z,\\
\psi_{2m}(z) &= a_{2m}\cosh z + b_{2m}\sinh z + c_{2m}z\cosh z + d_{2m}z\sinh z,\\
\psi_{3m}(z) &= a_{3m}\cosh z + b_{3m}\sinh z + c_{3m}z\cosh z + d_{3m}z\sinh z,\\
\psi_{4m}(z) &= a_{4m}\cosh z + b_{4m}\sinh z + c_{4m}z\cosh z + d_{4m}z\sinh z,
\end{aligned}\right\} \tag{6.97}$$

where for brevity we have introduced the notation

$$z=\frac{m\pi y}{l}; \tag{6.98}$$

the quantities a_{1m}, b_{1m}, ..., d_{4m} are so far arbitrary; we shall choose them later in such a manner as to satisfy the boundary conditions as simply as possible.

The general solution of equation (6.82), given by formula (6.85), is now replaced by

$$Y_m=\bar{A}_m\psi_{1m}(z)+\bar{B}_m\psi_{2m}(z)+\bar{C}_m\psi_{3m}(z)+\bar{D}_m\psi_{4m}(z), \tag{6.99}$$

where $\bar{A}_m$, $\bar{B}_m$, $\bar{C}_m$, $\bar{D}_m$ are new arbitrary constants. The stress function (6.88) becomes

$$\varphi=\sum_{m=1}^{n}\sin\frac{m\pi x}{l}Y_m. \tag{6.100}$$

In selecting the coefficients in formulas (6.97) the following considerations will be used as a guide. The components of the state of stress acting on the upper and lower faces of the strip are represented by

$$Y_y=\frac{\partial^2\varphi}{\partial x^2}=-\sum\left(\frac{m\pi}{l}\right)^2\sin\frac{m\pi x}{l}Y_m;$$

$$X_y=-\sum\frac{m\pi}{l}\cos\frac{m\pi x}{l}Y'_m.$$

There appear here functions (6.99) and their first derivatives; let us adjust functions (6.97) so that for $y=0$ and $y=h$ the

expressions of Y_m and Y'_m will be monomials; for this purpose we determine the coefficients of functions (6.97) from the following boundary conditions $\left(\text{where } \beta = \pi \frac{h}{l}\right)$:

$$\left.\begin{array}{l} \psi_{1m}(0)=1;\ \psi'_{1m}(0)=0;\ \psi_{1m}(m\beta)=0;\ \psi'_{1m}(m\beta)=0; \\ \psi_{2m}(0)=0;\ \psi'_{2m}(0)=1;\ \psi_{2m}(m\beta)=0;\ \psi'_{2m}(m\beta)=0; \\ \psi_{3m}(0)=0;\ \psi'_{3m}(0)=0;\ \psi_{3m}(m\beta)=1;\ \psi'_{3m}(m\beta)=0; \\ \psi_{4m}(0)=0;\ \psi'_{4m}(0)=0;\ \psi_{4m}(m\beta)=0;\ \psi'_{4m}(m\beta)=1. \end{array}\right\} \quad (6.101)$$

Here $\psi'_{km}(0)$ and $\psi'_{km}(m\beta)$ denote the derivatives of functions (6.97) with respect to the *argument*, with the substitution of its boundary values $z=0$ and $z=m\beta=\frac{m\pi h}{l}$. Introducingthe expressions of functions (6.97) in these conditions, we obtain four systems of equations for determining the coefficients

$$a_{km},\ b_{km},\ c_{km},\ d_{km},$$

where the index k is given the values 1, 2, 3, 4 in succession. These equations can readily be solved in general form; the following table presents the expressions of the coefficients thus obtained:

	a_{km}	b_{km}	c_{km}	d_{km}	
ψ_{1m}	1	$-\frac{\sinh m\beta \cosh m\beta + m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$\frac{\sinh m\beta \cosh m\beta + m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$-\frac{\sinh^2 m\beta}{\sinh^2 m\beta - (m\beta)^2}$	ψ_{1m}
ψ_{2m}	0	$-\frac{(m\beta)^2}{\sinh^2 m\beta - (m\beta)^2}$	$\frac{\sinh^2 m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$-\frac{\sinh m\beta \cosh m\beta - m\beta}{\sinh^2 m\beta - (m\beta)^2}$	ψ_{2m}
ψ_{3m}	0	$\frac{m\beta \cosh m\beta + \sinh m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$-\frac{m\beta \cosh m\beta + \sinh m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$\frac{m\beta \sinh m\beta}{\sinh^2 m\beta - (m\beta)^2}$	ψ_{3m}
ψ_{4m}	0	$-\frac{m\beta \sinh m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$\frac{m\beta \sinh m\beta}{\sinh^2 m\beta - (m\beta)^2}$	$-\frac{m\beta \cosh m\beta - \sinh m\beta}{\sinh^2 m\beta - (m\beta)^2}$	ψ_{4m}

These coefficients are independent of the load and can be calculated once and for ever for strips with a given ratio of the dimensions $h:l$ entering in the parameter $\beta=\pi\frac{h}{l}$; calculations are simplified by a pairwise equality of a series of coefficients

$$c_{1m} = -b_{1m},\quad c_{2m} = -d_{1m},\quad c_{3m} = -b_{3m},\quad c_{4m} = d_{3m} = -b_{4m}.$$

The particular solutions (6.97) of equation (6.82) thus constructed permit us to obtain the following simple boundary values of the functions $Y_m(y)$ and $Y'_m(y)$ at $y=0$ and $y=h$ (in differentiating we take into account that $\frac{dY_m}{dy}=\frac{dY_m}{dz}\frac{m\pi}{l}$):

$$\left.\begin{aligned} Y_m(0)&=\bar{A}_m; \quad Y'_m(0)=\frac{m\pi}{l}\bar{B}_m; \\ Y_m(h)&=\bar{C}_m; \quad Y'_m(h)=\frac{m\pi}{l}\bar{D}_m. \end{aligned}\right\} \tag{6.102}$$

This, in turn, results in a considerable simplification in the boundary conditions for a strip in comparison with those previously obtained (6.89). In fact, if the lower face ($y=0$) is acted upon by the loads $q_2(x)$ and $t_2(x)$ (Fig. 60), the corresponding boundary conditions on the basis of (6.102) will be written as

$$\left.\begin{aligned} Y_{-y}&=-\frac{\partial^2\varphi}{\partial x^2}=\sum\left(\frac{m\pi}{l}\right)^2\sin\frac{m\pi x}{l}Y_m(0)= \\ &\qquad=\sum\left(\frac{m\pi}{l}\right)^2\bar{A}_m\sin\frac{m\pi x}{l}=q_2(x), \\ -X_y&=\frac{\partial^2\varphi}{\partial x\,\partial y}=\sum\left(\frac{m\pi}{l}\right)^2\cos\frac{m\pi x}{l}Y'_m(0)= \\ &\qquad=\sum\left(\frac{m\pi}{l}\right)^2\bar{B}_m\cos\frac{m\pi x}{l}=t_2(x); \\ &\text{the conditions on the upper face } y=h \text{ will be} \\ Y_y&=\frac{\partial^2\varphi}{\partial x^2}=-\sum\left(\frac{m\pi}{l}\right)^2\sin\frac{m\pi x}{l}Y_m(h)= \\ &\qquad=-\sum\left(\frac{m\pi}{l}\right)^2\bar{C}_m\sin\frac{m\pi x}{l}=q_1(x), \\ X_y&=-\frac{\partial^2\varphi}{\partial x\,\partial y}=\sum\left(\frac{m\pi}{l}\right)^2\cos\frac{m\pi x}{l}Y'_m(h)= \\ &\qquad=-\sum\left(\frac{m\pi}{l}\right)^2\bar{D}_m\cos\frac{m\pi x}{l}=t_1(x). \end{aligned}\right\} \tag{6.103}$$

Let us expand the prescribed loads in trigonometric series:

$$\left.\begin{aligned} q_2(x)&=\sum_{m=1}^{n}A_m\sin\frac{m\pi x}{l}; \quad t_2=\sum_{m=1}^{n}B_m\cos\frac{m\pi x}{l}; \\ q_1(x)&=\sum_{m=1}^{n}C_m\sin\frac{m\pi x}{l}; \quad t_1=\sum_{m=1}^{n}D_m\cos\frac{m\pi x}{l}, \end{aligned}\right\} \tag{6.104}$$

in which the coefficients are found in accordance with Euler's formulas:

$$\left.\begin{aligned} A_m &= \frac{2}{l}\int_0^l q_2(x)\sin\frac{m\pi x}{l}\,dx; \quad B_m = \frac{2}{l}\int_0^l t_2(x)\cos\frac{m\pi x}{l}\,dx, \\ C_m &= \frac{2}{l}\int_0^l q_1(x)\sin\frac{m\pi x}{l}\,dx; \quad D_m = \frac{2}{l}\int_0^l t_1(x)\cos\frac{m\pi x}{l}\,dx. \end{aligned}\right\} \quad (6.105)$$

Substituting the expressions of the loads (6.104) in (6.103) and comparing the coefficients in the right-hand and left-hand members, we obtain

$$\left.\begin{aligned} \bar{A}_m &= \left(\frac{l}{m\pi}\right)^2 A_m; \qquad \bar{B}_m = \left(\frac{l}{m\pi}\right)^2 B_m; \\ \bar{C}_m &= -\left(\frac{l}{m\pi}\right)^2 C_m; \quad \bar{D}_m = -\left(\frac{l}{m\pi}\right)^2 D_m. \end{aligned}\right\} \quad (6.106)$$

Hence, the coefficients of functions (6.99) turn out to be proportional to the coefficients of the expansion of loads (6.105) and their determination does not require the solution of a system of equations [cf. equations (6.90b)]. The stress function (6.100) takes the final form satisfying the conditions on the upper and lower faces:

$$\varphi(x, y) = \sum_{m=1}^{n}\left(\frac{l}{m\pi}\right)^2 \sin\frac{m\pi x}{l} \times$$
$$\times\left[A_m\psi_{1m}(z) + B_m\psi_{2m}(z) + C_m\psi_{3m}(z) + D_m\psi_{4m}(z)\right], \quad (6.107)$$
$$z = \frac{m\pi y}{l}.$$

The stresses are calculated by the usual formulas:

$$X_x = \frac{\partial^2\varphi}{\partial y^2} = \sum \sin\frac{m\pi x}{l} Y''_m(y),$$
$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = -\sum\left(\frac{m\pi}{l}\right)^2 \sin\left(\frac{m\pi x}{l}\right) Y_m(y),$$
$$X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} = -\sum\left(\frac{m\pi}{l}\right)\cos\left(\frac{m\pi x}{l}\right) Y'_m(y).$$

Introducing here the expression of function (6.99), we obtain the final formulas of stresses for a strip under given loads on its long sides (in differentiation, as mentioned before, we take

into account that $\frac{d\psi_{km}(z)}{dy} = \frac{d\psi_{km}(z)}{dz}\frac{m\pi}{l}\Big)$:

$$\left.\begin{aligned}
X_x &= \sum \left[A_m\psi''_{1m}(z) + B_m\psi''_{2m}(z) - \right.\\
&\qquad \left. - C_m\psi''_{3m}(z) - D_m\psi''_{4m}(z)\right]\sin\frac{m\pi x}{l},\\
Y_y &= -\sum \left[A_m\psi_{1m}(z) + B_m\psi_{2m}(z) - \right.\\
&\qquad \left. - C_m\psi_{3m}(z) - D_m\psi_{4m}(z)\right]\sin\frac{m\pi x}{l},\\
X_y = Y_x &= -\sum \left[A_m\psi'_{1m}(z) + B_m\psi'_{2m}(z) - \right.\\
&\qquad \left. - C_m\psi'_{3m}(z) - D_m\psi'_{4m}(z)\right]\cos\frac{m\pi x}{l}.
\end{aligned}\right\} \quad (6.108)$$

In the design of a given strip under various loads, one has only to substitute the corresponding coefficients of expansion (6.105) in formulas (6.108); the functions $\psi_{km}(z)$, and their derivatives, however, are independent of loads. In most practical problems not all the loads allowed for in formulas (6.108) are involved, and the latter are simplified. For instance, in the case of a strip supported at the ends and loaded over the upper face (Fig. 61) one must set $A_m = B_m = D_m = 0$.

The foregoing method of solving the problem possesses certain advantages in cases where it is necessary to determine the displacements of the points of a strip u and v, which is required, for instance, in the solution of contact problems. The displacements are found in the same way as in the case of the cantilever considered in Sec. 41. Without presenting the appropriate course of reasoning (see the work cited in the footnote at the beginning of this section), we give some final results. The general formula of the vertical component of displacement has the form

$$\begin{aligned}
v = &-\frac{l}{E\pi}\sum\frac{1}{m}\left\{A_m\left[\Psi_{1m}(z) + \sigma\psi'_{1m}(z)\right] + B_m\left[\Psi_{2m}(z) + \sigma\psi'_{2m}(z)\right] - \right.\\
&\left. - C_m\left[\Psi_{3m}(z) + \sigma\psi'_{3m}(z)\right] - D_m\left[\Psi_{4m}(z) + \sigma\psi'_{4m}(z)\right]\right\}\sin\frac{m\pi x}{l} - \frac{\gamma x - \varepsilon}{E},
\end{aligned} \quad (6.109)$$

where $\Psi_{km}(z)$ $(k = 1, 2, 3, 4)$ denote the following antiderivatives of the functions $\psi_{km}(z)$ [see (6.97)]:

$$\begin{aligned}
\Psi_{km}(z) = \frac{l}{m\pi}\left[(b_{km} - c_{km})\cosh z + (a_{km} + d_{km})\sinh z + \right.\\
\left. + d_{km} z\cosh z + c_{km} z\sinh z\right];
\end{aligned}$$

γ is the angle of rotation and ε is the vertical component of rigid body displacement of the strip. Let us confine ourselves to the case of vertical loads $q_1(x)$ and $q_2(x)$ and, therefore, assume subsequently that $B_m = D_m = 0$. The displacements of the upper and lower faces of the strip are of prime interest in practical application; the following table gives the boundary values of the functions $\Psi_{km}(z)$ entering in formula (6.109):

	Ψ_{1m}	Ψ_{2m}	Ψ_{3m}	Ψ_{4m}
$z=0$	$-2c_{1m}$	$1-2c_{2m}$	$-2c_{3m}$	$-2c_{4m}$
$z=m\beta$	$2c_{3m}$	$-2c_{4m}$	$2c_{1m}$	$1-2c_{2m}$

By using them, we can easily find the displacement (deflection) of the lower face ($z=0$):

$$v = \frac{2l}{E\pi} \sum \frac{1}{m} [A_m c_{1m} - C_m c_{3m}] \sin \frac{m\pi x}{l} - \frac{\gamma x - \varepsilon}{E}. \qquad (6.110)$$

If the load acts only on the upper face, we have

$$v = -\frac{2l}{E\pi} \sum \frac{1}{m} C_m c_{3m} \sin \frac{m\pi x}{l} - \frac{\gamma x - \varepsilon}{E}. \qquad (6.110a)$$

With formula (6.109) it is possible to solve the problem of a strip resting on an absolutely rigid foundation and subjected to the load $q_2(x)$ over the upper face [formulas (6.104)]. In this case we have the mixed problem (Sec. 37). If there is no friction between the strip and the foundation, the conditions on the lower face will be: at $z=0$, $v=0$, $X_y=0$; it is assumed that the strip makes full contact with the foundation. The solution of the problem leads to the following expressions for stresses:

$$\left.\begin{aligned} X_x &= \sum C_m \left[\frac{c_{3m}}{c_{1m}} \psi''_{1m}(z) - \psi''_{3m}(z)\right] \sin \frac{m\pi x}{l}, \\ Y_y &= -\sum C_m \left[\frac{c_{3m}}{c_{1m}} \psi_{1m}(z) - \psi_{3m}(z)\right] \sin \frac{m\pi x}{l}, \\ X_y &= -\sum C_m \left[\frac{c_{3m}}{c_{1m}} \psi'_{1m}(z) - \psi'_{3m}(z)\right] \cos \frac{m\pi x}{l}. \end{aligned}\right\} \qquad (6.111)$$

Setting $z=0$ in the second formula, we find the reaction which the foundation exerts on the strip:

$$Y_y = -\sum C_m \frac{c_{3m}}{c_{1m}} \sin\frac{m\pi x}{l}. \tag{6.112}$$

The displacement (settlement) of the upper face under the action of the load is given by

$$v = -\frac{2l}{E\pi}\sum c_{1m}\frac{C_m}{m}\left[\left(\frac{c_{3m}}{c_{1m}}\right)^2 - 1\right]\sin\frac{m\pi x}{l}. \tag{6.113}$$

The simplicity of the last formulas shows that the foregoing method is practicable.

46. STRIP OF INFINITE LENGTH

The preceding sections have been concerned with a strip of finite length l; at the same time it may be said that we deal with an infinite strip, but one subjected to loads repeated periodically in the portions of length l since the whole solution is expressed in terms of periodic functions $\cos\frac{m\pi x}{l}$ and $\sin\frac{m\pi x}{l}$. it develops that it is possible to modify the solution obtained and to extend it to the problem for an infinite strip under an arbitrary load. With that end in view, let us return to the initial particular solution (6.86) for the stress function $\varphi(x, y)$. By assigning *discrete* values $\lambda=\frac{m\pi}{l}$ $(m=1, 2, \ldots, \infty)$ to the parameter λ, we obtain the solution of the problem in trigonometric series. Now we shall vary λ continuously from $-\infty$ to $+\infty$; this will lead to the solution of the problem in integrals. The coefficients A, B, C, D will also be considered as functions of λ; by setting the coefficients K_1 and K_2 equal to unity, without loss in generality, we write the solution (6.86) as

$$\varphi(x, y) = \sin\lambda x Y_1(\lambda y) + \cos\lambda x Y_2(\lambda y), \tag{6.114}$$

where

$$\left.\begin{aligned} Y_1(\lambda y) &= A(\lambda)\cosh\lambda y + B(\lambda)\sinh\lambda y + \\ &\quad + C(\lambda)\lambda y\cosh\lambda y + D(\lambda)\lambda y\sinh\lambda y, \\ Y_2(\lambda y) &= A_1(\lambda)\cosh\lambda y + B_1(\lambda)\sinh\lambda y + \\ &\quad + C(\lambda)\lambda y\cosh\lambda y + D(\lambda)\lambda y\sinh\lambda y. \end{aligned}\right\} \tag{6.115}$$

The sum of solutions (6.114) will now be represented in the form of the integral

$$\Phi(x, y) = \int_{-\infty}^{+\infty} \varphi\, d\lambda = \int_{-\infty}^{+\infty} \cos \lambda x Y_1(\lambda y)\, d\lambda + \int_{-\infty}^{+\infty} \sin \lambda x Y_2(\lambda y)\, dy; \tag{6.116}$$

obviously, it will also be a solution of equation (6.77). With the stress function (6.116) we obtain the stresses

$$\left.\begin{aligned}
X_x &= \frac{\partial^2 \Phi}{\partial y^2} = \int_{-\infty}^{+\infty} \lambda^2 \cos \lambda x Y_1''(\lambda y)\, d\lambda + \int_{-\infty}^{+\infty} \lambda^2 \sin \lambda x Y_2''(\lambda y)\, d\lambda,\\
Y_y &= \frac{\partial^2 \Phi}{\partial x^2} = -\int_{-\infty}^{+\infty} \lambda^2 \cos \lambda x Y_1(\lambda y)\, d\lambda - \int_{-\infty}^{+\infty} \lambda^2 \sin \lambda x Y_2(\lambda y)\, d\lambda,\\
X_y &= -\frac{\partial^2 \Phi}{\partial x\, \partial y} = \int_{-\infty}^{+\infty} \lambda^2 \sin \lambda x Y_1'(\lambda y)\, d\lambda - \int_{-\infty}^{+\infty} \lambda^2 \cos \lambda x Y_2'(\lambda y)\, d\lambda.
\end{aligned}\right\} \tag{6.117}$$

Here single and double primes denote the derivatives of the functions $Y_1(\lambda y)$ and $Y_2(\lambda y)$ with respect to the argument λy.

If loads $q_1(x)$, $t_1(x)$, $q_2(x)$, $t_2(x)$ are given on the upper and lower faces (Fig. 60), the boundary conditions will be of the form similar to (6.90a) (the limits of integration are omitted for simplicity in writing):

$$\left.\begin{aligned}
&\text{for } y = 0:\\
&Y_{-y} = q_2(x) = \int \lambda^2 \cos \lambda x Y_1(0)\, d\lambda + \int \lambda^2 \sin \lambda x Y_2(0)\, d\lambda,\\
&X_{-y} = t_2(x) = -\int \lambda^2 \sin \lambda x Y_1'(0)\, d\lambda + \int \lambda^2 \cos \lambda x Y_2'(0)\, d\lambda;\\
&\text{for } y = h:\\
&Y_y = q_1(x) = -\int \lambda^2 \cos \lambda x Y_1(\lambda h)\, d\lambda - \int \lambda^2 \sin \lambda x Y_2(\lambda h)\, d\lambda,\\
&X_y = t_1(x) = \int \lambda^2 \sin \lambda x Y_1'(\lambda h)\, d\lambda - \int \lambda^2 \cos \lambda x Y_2'(\lambda h)\, d\lambda.
\end{aligned}\right\} \tag{6.118}$$

These conditions will be used as a basis for determining the coefficients of functions (6.115):

$$\left.\begin{matrix}
A(\lambda), & B(\lambda), & C(\lambda), & D(\lambda);\\
A_1(\lambda), & B_1(\lambda), & C(\lambda), & D(\lambda).
\end{matrix}\right\} \tag{6.119}$$

The boundary values of these functions enter in the integrands of (6.118); they are to be found in the first place. In solving the problem by means of series such an operation amounts to determination of coefficients according to Euler's formulas; in the method under consideration, this requires the use of *Fourier's formula* expressing an arbitrary function $f(x)$ (satisfying the so-called Dirichlet conditions) in the form of a double integral with the integrand containing the same function $f(x)$. The derivation of this formula may be found in the courses on integral calculus; it has the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\alpha)\cos\lambda(\alpha-x)\,d\alpha\right]d\lambda. \qquad (6.120)$$

Here x plays the role of a parameter; the inner integration is carried out with respect to the variable α, and the outer—with respect to λ. Writing out the expression of the cosine, we put (6.120) in the following form (the limits of integration are omitted as before):

$$f(x)=\frac{1}{2\pi}\int \cos\lambda x\left[\int f(\alpha)\cos\lambda\alpha\,d\alpha\right]d\lambda+$$
$$+\frac{1}{2\pi}\int \sin\lambda x\left[\int f(\alpha)\sin\lambda\alpha\,d\alpha\right]d\lambda. \qquad (6.120a)$$

Its structure is analogous to that of formulas (6.118), and from their comparison we obtain the desired boundary values of functions Y_1 and Y_2 and their derivatives. Indeed, assuming that $f(x)=q_1(x)$ in (6.120a) and comparing the result with the third of formulas (6.118), we find

$$Y_1(\lambda h)=-\frac{1}{2\pi\lambda^2}\int_{-\infty}^{+\infty} q_1(\alpha)\cos\lambda\alpha\,d\alpha;\quad Y_2(\lambda h)=-\frac{1}{2\pi\lambda^2}\int_{-\infty}^{+\infty} q_1(\alpha)\sin\lambda\alpha\,d\alpha.$$

Setting now $f(x)=t_1(x)$ and comparing with the fourth formula of (6.118), we obtain

$$Y_1'(\lambda h)=\frac{1}{2\pi\lambda^2}\int_{-\infty}^{+\infty} t_1(\alpha)\sin\lambda\alpha\,d\alpha;\quad Y_2'(\lambda h)=\frac{1}{2\pi\lambda^2}\int_{-\infty}^{+\infty} t_1(\alpha)\cos\lambda\alpha\,d\alpha.$$

In the same way, by the use of the first two formulas of (6.118) we find four boundary values of the functions

$$Y_1(0),\quad Y_2(0),\quad Y_1'(0),\quad Y_2'(0).$$

The reader is recommended to derive these formulas as an exercise. These eight boundary values of functions (6.115) and their derivatives must be substituted in the left-hand members of these equalities setting $y=0$ or $y=h$, respectively, in the right-hand members. This will give eight equations for the determination of coefficients (6.119), and the boundary conditions for the strip will be satisfied; it remains to determine the stresses according to formulas (6.117).

The presence in the solution of integrals with infinite limits requires certain restrictions on loads if they are not applied in a portion of finite length but continue infinitely: in this case their resultants must be finite; otherwise the integrals may loose sense (become infinite). The integrals entering in the solution are taken in finite form only in rare cases; one has to evaluate them by the numerical method.

VII

Plane Problem in Polar Co-ordinates

47. GENERAL EQUATIONS OF THE PLANE PROBLEM IN POLAR CO-ORDINATES

Up to this point, in solving the problems of the theory of elasticity we have used the Cartesian co-ordinates in which a point M (x_0, y_0, z_0) is located by the intersection of three planes:

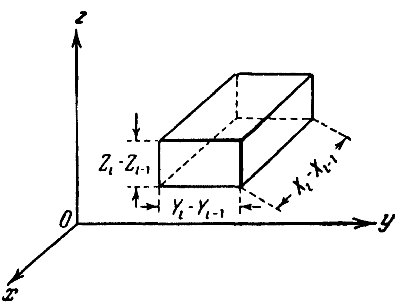

Fig. 63

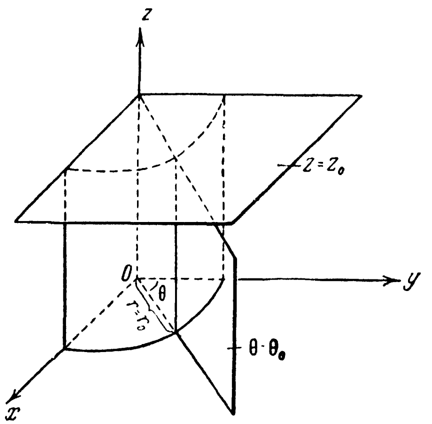

Fig. 64

$x = x_0$; $y = y_0$; $z = z_0$. Applying these co-ordinates we divide the body under investigation into infinitesimal elements by three systems of planes:

$$x = x_1; \quad x = x_2; \quad \ldots; \quad x = x_i;$$
$$y = y_1; \quad y = y_2; \quad \ldots; \quad y = y_i;$$
$$z = z_1; \quad z = z_2; \quad \ldots; \quad z = z_i.$$

The surface on which one of the co-ordinates retains the constant value is called a co-ordinate surface; in our case these surfaces are planes parallel to the co-ordinate planes, the resulting elements being of the form of parallelepipeds (Fig. 63).

In many problems, however, it is found more convenient to select other systems of co-ordinates. In particular, consider cylindrical co-ordinates in which a point (Fig. 64) is located by three numbers: r, θ, z. In other words, a point is located by the intersection of the following co-ordinate surfaces: a circular cylinder $r=r_0$, a plane $\theta=\theta_0$ passing through the axis Oz and a plane $z=z_0$ parallel to Oxy. Accordingly, in this case we divide the body under investigation into infinitesimal elements by the following three systems of surfaces: a system of concentric cylinders $r=r_1$, $r=r_2$, ..., $r=r_i$; a set of planes $\theta=\theta_1$, $\theta=\theta_2$, ...,

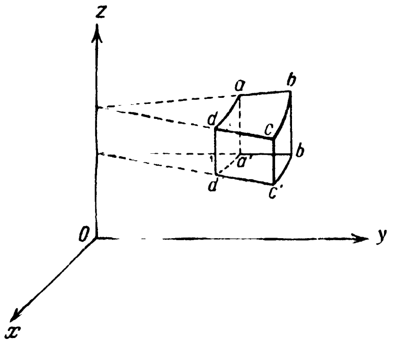

Fig. 65

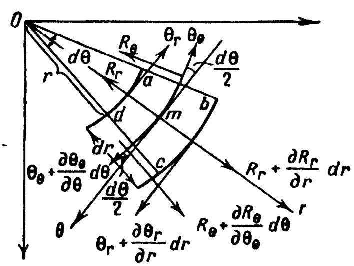

Fig. 66

$\theta=\theta_i$ containing the axis Oz, and a system of planes $z=z_1$, $z=z_2$, ..., $z=z_i$ parallel to Oxy. The form of the resulting infinitesimal element is shown in Fig. 65.

Let us apply the cylindrical co-ordinates to the plane problem, the axis of the prismatical body under investigation being parallel to the axis Oz. The external load, according to our assumptions, will be parallel to the plane Oxy. In this case, as we have seen, both for plane strain and for plane stress the axis Oz may be removed formally with the result that the whole problem will be solved as if in the plane, in polar co-ordinates r, θ.

Let us now derive the basic equations of the theory of elasticity for the plane problem in polar co-ordinates. First we take the differential equations of equilibrium (I). We isolate from the body an element *abcd* with central angle $d\theta$ and the smallest radius r. Its sides will be (Fig. 66)

$$\left.\begin{aligned} ab &= cd = dr, \\ ad &= r\,d\theta, \\ bc &= (r+dr)\,d\theta. \end{aligned}\right\} \tag{7.1}$$

We select two axes, mr and $m\theta$, and by using them designate the stresses acting upon the faces of the element according to the rule accepted at the beginning; these notations are shown in the drawing. We set up the conditions of equilibrium of the element $abcd$ by projecting the forces applied to it on the axes r and θ. The body forces are neglected. The thickness of the element along the axis Oz is taken to be unity. In projecting the forces, in view of the infinite smallness of the angle $d\theta$, we shall assume

$$\sin\frac{d\theta}{2}=\frac{d\theta}{2}, \quad \cos\frac{d\theta}{2}=1.$$

Multiplying the stresses, indicated in the drawing, by the corresponding areas (7.1) and projecting the resulting forces on the axes r and θ, we find

$$\left(R_r+\frac{\partial R_r}{\partial r}dr\right)(r+dr)\,d\theta-R_r r\,d\theta+\left(R_\theta+\frac{\partial R_\theta}{d\theta}d\theta\right)dr-$$
$$-R_\theta\,dr-\left(\Theta_\theta+\frac{\partial\Theta_\theta}{\partial\theta}d\theta\right)dr\frac{d\theta}{2}-\Theta_\theta\,dr\frac{d\theta}{2}=0,$$

$$\left(\Theta_\theta+\frac{\partial\Theta_\theta}{\partial\theta}d\theta\right)dr-\Theta_\theta\,dr+\left(R_\theta+\frac{\partial R_\theta}{\partial\theta}d\theta\right)dr\frac{d\theta}{2}+$$
$$+R_\theta\,dr\frac{d\theta}{2}+\left(\Theta_r+\frac{\partial\Theta_r}{\partial r}dr\right)(r+dr)\,d\theta-\Theta_r r\,d\theta=0.$$

Cancelling out the obtained equations and disregarding infinitesimal quantities of the third order, we get

$$R_r\,dr\,d\theta+\frac{\partial R_r}{\partial r}r\,dr\,d\theta+\frac{\partial R_\theta}{\partial\theta}d\theta\,dr-2\Theta_\theta\frac{dr\,d\theta}{2}=0,$$

$$\frac{\partial\Theta_\theta}{\partial\theta}d\theta\,dr+2R_\theta\frac{dr\,d\theta}{2}+\Theta_r\,dr\,d\theta+\frac{\partial\Theta_r}{\partial r}r\,dr\,d\theta=0.$$

Further, dividing these equations by $dr\,d\theta$ and bearing in mind that $\Theta_r=R_\theta$, we shall have the final differential equations of equilibrium in polar co-ordinates:

$$\left.\begin{aligned}\frac{\partial R_r}{\partial r}+\frac{1}{r}\frac{\partial R_\theta}{\partial\theta}+\frac{R_r-\Theta_\theta}{r}=0,\\ \frac{\partial R_\theta}{\partial r}+\frac{1}{r}\frac{\partial\Theta_\theta}{\partial\theta}+\frac{2R_\theta}{r}=0.\end{aligned}\right\}\qquad\text{(Ipp)}$$

For the equations of equilibrium in Cartesian co-ordinates (6.14) corresponding to (Ipp), it has proved possible to find the general solution (6.16) expressed through the stress function $\varphi(x, y)$. An analogous solution can be obtained for equations (Ipp) by the use of the stress function $\varphi(r, \theta)$; there are several methods of deriving this solution; however, we shall not present them

here; we write only the result:

$$\left.\begin{aligned} R_r &= \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}, \\ \Theta_\theta &= \frac{\partial^2\varphi}{\partial r^2}, \\ R_\theta &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = -\frac{1}{r}\frac{\partial^2\varphi}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}. \end{aligned}\right\} \qquad \text{(VIIIp)}$$

By substituting these expressions in equations (Ipp), we see that they are satisfied identically. It is necessary, of course, to add to them the condition of compatibility of strain components; in Cartesian co-ordinates it has the form (IVp, Sec. 39)

$$\nabla^2 (X_x + Y_y) = 0. \qquad \text{(IVp)}$$

It is well to recall now that

$$\Theta = X_x + Y_y + Z_z$$

represents an invariant under a transformation of co-ordinates and, therefore, its magnitude will not change if we replace the Cartesian co-ordinates by the polar co-ordinates. In the case of plane strain ($w = 0$)

$$\Theta = X_x + Y_y - \sigma (X_x + Y_y) = (1 - \sigma)(X_x + Y_y).$$

In the case of generalised plane stress ($Z_z = 0$)

$$\Theta = X_x + Y_y.$$

Thus, $X_x + Y_y$ will be an invariant in the plane problem and, therefore,

$$X_x + Y_y \equiv R_r + \Theta_\theta.$$

On this basis we obtain directly from (IVp) the compatibility condition in polar co-ordinates:

$$\nabla^2 (R_r + \Theta_\theta) = 0. \qquad (7.2)$$

However, the invariant $X_x + Y_y$ represents the Laplacian operator on the stress function in Cartesian co-ordinates:

$$X_x + Y_y = \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial x^2} = \nabla^2\varphi(x, y);$$

consequently, on the basis of (VIIIp)

$$R_r + \Theta_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial r^2} = \nabla^2\varphi(r, \theta)$$

is the Laplacian operator on the same function in polar co-ordinates; hence the general conclusion: the symbol of the Lapla-

cian operator in polar co-ordinates implies the following differential operation:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2}. \tag{7.3}$$

Consequently, the compatibility equation (7.2) in polar co-ordinates has the form

$$\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial r^2}\right) = 0. \tag{IXp}$$

Carrying out the operations indicated in the first parentheses, we obtain the partial differential equation of the fourth order

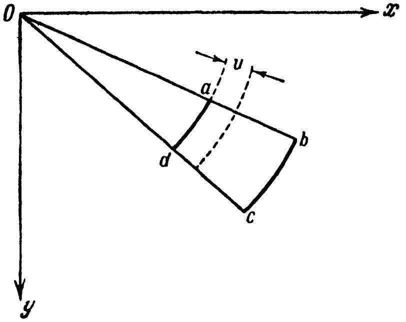

Fig. 67

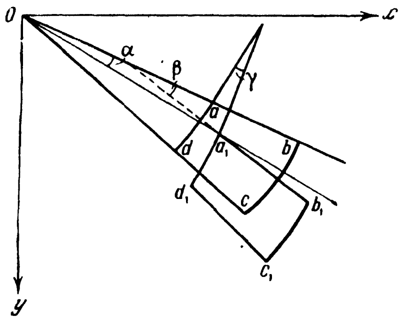

Fig. 68

which must be satisfied by the stress function $\varphi(r, \theta)$; by the use of it we calculate the stresses according to formulas (VIIIp).

We now fix attention on deformations and displacements (Fig. 67). Let us designate the displacements of points along the r axis by u, and along the θ axis by v; the elongation along the r axis by e_{rr}; along the θ axis by $e_{\theta\theta}$ and shear, i.e., the distortion of the right angle bad, by $e_{r\theta}$.

By the same reasoning as before (Sec. 10), we obtain $e_{rr} = \frac{\partial u}{\partial r}$. We proceed to the elongation $e_{\theta\theta}$. This elongation may be due to two causes. First, if only radial displacements u occur, the unit elongation of the linear element $ad = r\,d\theta$ is given by

$$\frac{(r+u)\,d\theta - r\,d\theta}{r\,d\theta} = \frac{u}{r}.$$

Due to displacements v along the θ axis there also results the unit elongation $\frac{\partial v}{\partial s} = \frac{\partial v}{r\,\partial \theta}$; the total elongation is

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}.$$

Let us now examine the shear. One sees from the drawing (Fig. 68) that the shear is expressed as

$$e_{r\theta} = (\beta - \alpha) + \gamma; \tag{7.4}$$

next we note that

$$\beta = \frac{\partial v}{\partial r}; \quad \alpha = \frac{v}{r}; \quad \gamma = \frac{\partial u}{\partial s} = \frac{\partial u}{r\,\partial\theta}.$$

Substituting these values in formula (7.4), we obtain

$$e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Thus, in place of equations (III′p) for the plane problem we now obtain the following system of equations:

$$\left.\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \\ e_{r\theta} &= \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}. \end{aligned}\right\} \tag{IIIpp}$$

Hooke's law (V′p) for the case of generalised plane stress has the previous form, only the notations of stresses and strains vary:

$$\left.\begin{aligned} e_{rr} &= \frac{1}{E}(R_r - \sigma\Theta_\theta), \\ e_{\theta\theta} &= \frac{1}{E}(\Theta_\theta - \sigma R_r), \\ e_{r\theta} &= \frac{2(1+\sigma)}{E} R_\theta. \end{aligned}\right\} \tag{Vpp}$$

In the case of plane strain E and σ should be replaced by E_1 and σ_1 according to formulas (6.5).

48. PROBLEMS IN WHICH STRESSES ARE INDEPENDENT OF THE POLAR ANGLE

Let us apply the equations derived to the solution of some problems. The problem is simplest to solve when stresses (VIIIp) are independent of the angle θ, i.e., when R_r, Θ_θ and R_θ are the same at all points of any circle with centre at the pole O.

The stress function φ can obviously be assumed to be independent of θ in this case; equation (IXp) is then simplified:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) = 0.$$

Carrying out differentiation, we obtain

$$\frac{\partial^4\varphi}{\partial r^4}+\frac{2}{r}\frac{\partial^3\varphi}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^3}\frac{\partial\varphi}{\partial r}=0; \tag{IX'p}$$

the expressions of stresses (VIIIp) become

$$R_r=\frac{1}{r}\frac{\partial\varphi}{\partial r};\quad \Theta_\theta=\frac{\partial^2\varphi}{\partial r^2};\quad R_\theta=\Theta_r=0. \tag{VIII'p}$$

Equation (IX′p) can easily be integrated by the use of the substitution $\varphi=r^n$ which leads to the characteristic equation with two multiple roots. As a result, the general solution of equation (IXp) has the form

$$\varphi(r)=A\ln r+Br^2\ln r+Cr^2+D. \tag{7.5}$$

Hence we find stresses according to (VIIIp)

$$\left.\begin{aligned} R_r&=\frac{A}{r^2}+2B\ln r+C+2C,\\ \Theta_\theta&=-\frac{A}{r^2}+2B\ln r+3B+2C,\\ R_\theta&=\Theta_r=0. \end{aligned}\right\} \tag{7.6}$$

By way of example, we shall use equations (7.6) for the solution of Lamé's problem of a circular tube submitted to uniform external and internal compression. Let the external pressure be p_2, and internal p_1.

Then, we obviously obtain the following conditions on the surface (Fig. 69):

$$\left.\begin{aligned} &\text{for } r=a\quad R_{-r}=+p_1 \text{ or } R_r=-p_1,\\ &\text{for } r=b\quad R_r\ \ =-p_2. \end{aligned}\right\} \tag{7.7}$$

To obtain Lamé's solution in these conditions the arbitrary constant B is set equal to zero; then equations (7.6) give

$$R_r=\frac{A}{r^2}+2C;\quad \Theta_\theta=-\frac{A}{r^2}+2C. \tag{7.8}$$

The constants A and C can easily be found from conditions (7.7); by satisfying them we obtain

$$\left.\begin{aligned} R_r&=\frac{a^2b^2}{b^2-a^2}\,\frac{p_2-p_1}{r^2}+\frac{a^2p_1-b^2p_2}{b^2-a^2},\\ \Theta_\theta&=-\frac{a^2b^2}{b^2-a^2}\,\frac{p_2-p_1}{r^2}+\frac{a^2p_1-b^2p_2}{b^2-a^2}. \end{aligned}\right\} \tag{7.8a}$$

In the solution of this problem we have made an assumption by setting in advance $B=0$; with three arbitrary constants A, B and C and two conditions (7.7) we could solve the problem under other assumptions as well; we can prove, however, that Lamé's solution (7.8a) is the one corresponding to the actual stress distribution. A feature of this problem is the fact that we encounter here a *doubly connected contour* since the cross section of the tube is bounded by two closed curves which do not intersect each other; in the presence of a doubly connected or a multiply connected contour the problem becomes more involved, in general, and the solution may not be unique. This difficulty can be avoided by applying either of the following two methods.

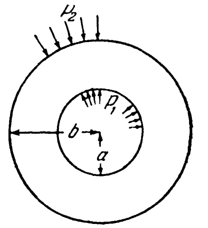

Fig. 69

First method. The solution of the problem in terms of displacements. Since stresses and strains are independent of the polar angle θ, there remains only the first of the equations of equilibrium (Ipp) in its simplified form

$$\frac{dR_r}{dr}+\frac{R_r-\Theta_\theta}{r}=0; \tag{7.9}$$

relations (IIIpp) are also simplified:

$$e_{rr}=\frac{du}{dr};\quad e_{\theta\theta}=\frac{u}{r};\quad e_{r\theta}=0. \tag{7.10}$$

Hooke's law (6.4) in the case of plane strain is

$$\left.\begin{aligned} e_{rr}&=\frac{1}{E_1}(R_r-\sigma_1\Theta_\theta),\\ e_{\theta\theta}&=\frac{1}{E_1}(\Theta_\theta-\sigma_1 R_r).\end{aligned}\right\} \tag{7.11}$$

Hence

$$R_r=\frac{E_1}{1-\sigma_1^2}(e_{rr}+\sigma_1 e_{\theta\theta});\quad \Theta_\theta=\frac{E_1}{1-\sigma_1^2}(e_{\theta\theta}+\sigma_1 e_{rr}), \tag{7.11a}$$

or, substituting here from (7.10), we have

$$R_r=\frac{E_1}{1-\sigma_1^2}\left(\frac{du}{dr}+\sigma_1\frac{u}{r}\right);\quad \Theta_\theta=\frac{E_1}{1-\sigma_1^2}\left(\frac{u}{r}+\sigma_1\frac{du}{dr}\right). \tag{7.12}$$

By introducing these expressions of stresses in the equation of equilibrium (7.9), we arrive at the differential equation of the second order with respect to u; its general solution will involve only two arbitrary constants, which are determined from the conditions on the outer and inner contours. Hence we obtain the

same formulas for stresses (7.8) from (7.12). This method is given in the courses on strength of materials.

Second method. The problem is solved, as before, in terms of stresses. The equation of equilibrium (7.9) is supplemented by the compatibility condition which is obtained by eliminating the only displacement u from relations (7.10). The second of these relations gives $u=re_{\theta\theta}$, hence $\frac{du}{dr}=e_{\theta\theta}+r\frac{de_{\theta\theta}}{dr}$. Substituting this in the first relation, we obtain the required compatibility condition in the form of a simple differential equation *of the first order*:

$$\frac{e_{rr}-e_{\theta\theta}}{r}=\frac{de_{\theta\theta}}{dr}. \tag{7.13}$$

Substituting here the values of strains from (7.11), we obtain the compatibility condition in terms of stresses:

$$\frac{d\Theta_\theta}{dr}-\sigma\frac{dR_r}{dr}=(1+\sigma_1)\frac{R_r-\Theta_\theta}{r}. \tag{7.13a}$$

Unlike (7.2), it turns out to be an equation of the first order as the axisymmetrical character of the problem has imposed certain restrictions on strains and stresses; the class of possible solutions of equation (7.13a) is considerably narrower than that of equation (7.2); it does not involve the solutions which do not correspond to the axisymmetrical problem and among these the solution corresponding to the value $B \neq 0$ in (7.5). Let us integrate the equilibrium equation (7.9) and the compatibility equation (7.13a); multiplying the former by $(1+\sigma_1)$ and adding it to the latter, we obtain

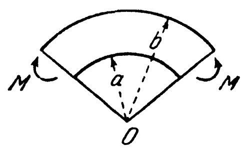

Fig. 70

$$\frac{d\Theta_\theta}{dr}+\frac{dR_r}{dr}=0,$$

whence

$$\Theta_\theta+R_r=2C_1 \quad \text{or} \quad \Theta_\theta=2C_1-R_r.$$

This enables us to eliminate Θ_θ from (7.9), and it becomes

$$\frac{dR_r}{dr}+\frac{2(R_r-C_1)}{r}=0.$$

Integrating it, we find

$$R_r=C_1+\frac{C_2}{r^2}; \quad \Theta_\theta=C_1-\frac{C_2}{r^2},$$

which coincides with (7.8).

Consider a part of a circular ring bent by forces applied to the end sections and reducing to couples (Fig. 70). The surface conditions will be expressed as

$$\text{for} \quad r=a \quad R_r=0;$$
$$\text{for} \quad r=b \quad R_r=0;$$
$$\int_a^b \Theta_\theta \, dr = 0; \quad \int_a^b \Theta_\theta r \, dr = M.$$

Writing out these conditions, we get

$$\frac{A}{a^2}+2B\ln a+B+2C=0; \quad \frac{A}{b^2}+2B\ln b+B+2C=0;$$
$$b\left(\frac{A}{b^2}+2B\ln b+B+2C\right)-a\left(\frac{A}{a^2}+2B\ln a+B+2C\right)=0;$$
$$-A\ln\frac{b}{a}+B(b^2\ln b-a^2\ln a)+(B+C)(b^2-a^2)=M.$$

We have obtained four equations for determining three constants A, B and C; we note, however, that the third equation is a consequence of the first two equations. Solving these equations, we find

$$A=\frac{4M}{N}a^2b^2\ln\frac{b}{a}; \quad B=\frac{2M}{N}(b^2-a^2);$$
$$C=-\frac{M}{N}[b^2-a^2+2(b^2\ln b-a^2\ln a)],$$

where

$$N=(b^2-a^2)^2-4a^2b^2\left(\ln\frac{b}{a}\right)^2.$$

Introducing these values in formulas (7.6), we obtain the following stresses:

$$\left.\begin{aligned} R_r &= \frac{4M}{N}\left(\frac{a^2b^2}{r^2}\ln\frac{b}{a}-a^2\ln\frac{r}{a}-b^2\ln\frac{b}{r}\right), \\ \Theta_\theta &= \frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\ln\frac{b}{a}-a^2\ln\frac{r}{a}-b^2\ln\frac{b}{r}-a^2+b^2\right), \\ \Theta_r &= 0. \end{aligned}\right\} \quad (7.14)$$

Comparing the stress Θ_θ as computed by formulas (7.14) with that given by the elementary theory of rods of large curvature (the hypothesis of plane sections), one can see that the difference obtained is not great; it must be attributed to the fact that the elementary theory does not take into account the stresses R_r caused by the mutual pressure of individual curvilinear longi-

tudinal fibres; these stresses produce additional deformation of a rod [1].

Next consider in this problem the change of the central angle $d\theta$ of an infinitesimal element *abcd* (Fig. 71). Since shear deformations are absent ($\Theta_r = 0$) in this case, the change of the angle $\delta(d\theta)$ depends on two elongations: e_{rr} and $e_{\theta\theta}$. Figs 71*b*

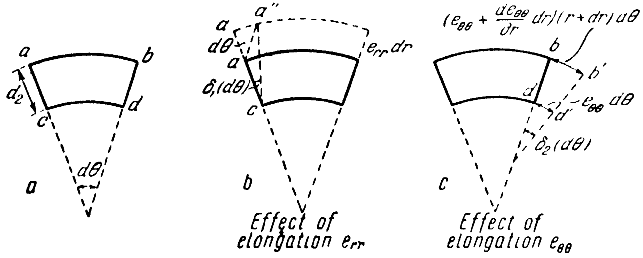

Fig. 71

and *c* show clearly the effect of both of these factors; we obtain from them the effect of the radial elongation e_{rr} (Fig. 71*b*):

$$\delta_1(d\theta) = -\frac{a'a''}{ca'} = -\frac{e_{rr}dr\,d\theta}{dr + e_{rr}\,dr} \approx -e_{rr}\,d\theta,$$

the effect of the axial elongation $e_{\theta\theta}$ (Fig. 71*c*)

$$\delta_2(d\theta) = \frac{bb' - dd'}{bd};$$

but

$$bb' - dd' = \left(e_{\theta\theta} + \frac{\partial e_{\theta\theta}}{\partial r}\,dr\right)(r + dr)\,d\theta - e_{\theta\theta}r\,d\theta \approx$$

$$\approx e_{\theta\theta}\,dr\,d\theta + r\frac{\partial e_{\theta\theta}}{\partial r}\,dr\,d\theta; \quad bd = dr;$$

therefore,

$$\delta_2(d\theta) \approx \left(e_{\theta\theta} + r\frac{\partial e_{\theta\theta}}{\partial r}\right)d\theta.$$

Thus, the total change of the angle $d\theta$ is

$$\delta(d\theta) = \delta_1(d\theta) + \delta_2(d\theta) = \left(e_{\theta\theta} + r\frac{\partial e_{\theta\theta}}{\partial r} - e_{rr}\right)d\theta,$$

and the unit change is

$$e_\theta = \frac{\delta(d\theta)}{d\theta} = e_{\theta\theta} + r\frac{\partial e_{\theta\theta}}{\partial r} - e_{rr}.$$

---

[1] The reader will notice that in a similar case of pure bending of a straight rod the stresses Y_y, corresponding to the stresses R_r of the above case, were absent [formula (6.28) of Sec. 40]; consequently, there is no mutual pressure of individual fibres in a straight rod under pure bending

If we substitute here in place of strains their expressions in terms of stresses from (Vp) of Sec. 47 and make use of equations (7.14), we can easily see that e_θ is independent of the radius r. This will evidently show that plane cross sections remain plane in the case of pure bending, thus supporting the hypothesis of plane sections which is usually adopted in the elementary theory of curved rods.

49. EFFECT OF A CONCENTRATED FORCE (FLAMANT-BOUSSINESQ PROBLEM)

Consider (Fig. 72) a homogeneous elastic medium bounded by a plane AB and extending indefinitely down from this plane. Let a force P be applied at a point C of the medium; with regard to the extension of the medium in the direction normal to the plane of the drawing there may be two cases as always in the plane problem: either this extension is very small (generalised plane stress) or it is unlimited (plane strain); in the latter case the force P must be assumed to be not concentrated but uniformly distributed along a straight line normal to the plane of the drawing and passing through the point C.

It is evident that the stresses will be very great and will exceed the elastic limit near the point C; we shall, therefore, consider only points lying outside a small region bounded by a circle mn of radius ρ; in view of this, by using Saint-Venant's principle we replace the load P by a load distributed along a semi-circumference mn and equivalent to the load P; this will have little effect on the stresses at points remote from the circle mn.

Let us attempt to satisfy the conditions of the problem by the following simple assumptions:

(1) there are no shearing stresses at any point M on the area normal to the radius $CM = r$;

(2) compressive normal stress R_r on this area is inversely proportional to the radius r and directly proportional to $\cos\theta$;

(3) normal stresses Θ_θ on areas in the radial direction are absent.

These assumptions lead to the following system of stresses:

$$\left.\begin{aligned} R_r &= -k\frac{\cos\theta}{r}, \\ \Theta_\theta &= 0, \\ R_\theta &= \Theta_r = 0. \end{aligned}\right\} \qquad (7.15)$$

These stresses satisfy the conditions of the problem on the surface AB; indeed, the bounding plane AB $\left(\theta = \pm\frac{\pi}{2},\ \cos\theta = 0\right)$ is free of stresses except for the point C where R_r assumes the indefinite form $\frac{0}{0}$; but the point C and even the small semicircle mn have been excluded from our analysis. Furthermore, the coefficient k in equation (7.15) can easily be selected so that the normal stresses R_r along the semicircumference mn will be equivalent to the given load P, i.e., so that the sum of the projections of the forces

$$R_r\rho\, d\theta$$

along the semicircumference $r = \rho$ on the axis Ox will be equal to P:

$$\int_{\frac{\pi}{2}}^{+\frac{\pi}{2}} R_{-r}\rho\cos\theta\, d\theta = P. \tag{7.16}$$

Substituting here the value R_r from (7.15) and carrying out integration, we easily obtain

$$k = \frac{2P}{\pi}. \tag{7.17}$$

The stress distribution as defined by formulas (7.15) is sometimes called a *simple radial distribution* of stresses.

In order to finally accept the above-constructed distribution of stresses (7.15) it is necessary to check whether it satisfies the differential equations of equilibrium (Ipp) and the equation of compatibility (IX′p). Since $\Theta_\theta = R_\theta = 0$, the equations of equilibrium reduce to a single equation

$$\frac{\partial R_r}{\partial r} + \frac{R}{r} = 0. \tag{7.18}$$

Substituting in it the value of R_r from (7.15), we see that it is satisfied identically. The equation of compatibility (IX′p) assumes a simple form

$$\nabla^2 R_r = \frac{1}{r}\frac{\partial R_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 R_r}{\partial\theta^2} + \frac{\partial^2 R_r}{\partial r^2} = 0.$$

Substitution of the value of R_r in it shows that it is also satisfied identically.

* It may be shown that stresses (7.15) are obtainable according to formulas (VIIIp) by the use of the stress function

$$\varphi = \frac{k}{2} r\theta \sin\theta. \tag{7.19}$$

In fact, these formulas, after the substitution of values (7.15), are transformed into a system of differential equations:

$$\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial r^2} = -k\frac{\cos\theta}{r};$$

$$\frac{\partial^2\varphi}{\partial r^2} = 0; \quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = 0.$$

The last two of these equations will be satisfied if we set $\varphi = rf(\theta)$. Then the first equation will be transformed into an ordinary differential equation

$$f''(\theta) + f(\theta) = -k\cos\theta.$$

Its general solution is

$$\varphi(\theta) = -\frac{k}{2}\theta\sin\theta + A\cos\theta + B\sin\theta,$$

and then

$$\varphi = rf(\theta) = -\frac{k}{2}r\theta\sin\theta + Ar\cos\theta + Br\sin\theta.$$

The last two terms can be disregarded since in Cartesian co-ordinates they represent a linear function

$$Ar\cos\theta + Br\sin\theta = Ax + By,$$

which has no influence on the stresses.

We have proved that stresses (7.15) satisfy all equations of the theory of elasticity: on the basis of (7.17) they are expressed as

$$\left.\begin{aligned} R_r &= -\frac{2P}{\pi}\frac{\cos\theta}{r}, \\ \Theta_\theta &= 0, \quad R_\theta = 0. \end{aligned}\right\} \tag{7.20}$$

Their investigation leads us to the following conclusions:

(1) the area at any point M (Fig. 72) which is normal to the radius r is principal since there are no shearing stresses on it;

(2) let us construct a circle CMx (Fig. 72) passing through the given point M and let its diameter be $Cx = d$; then

$$CM = Cx\cos\theta;$$

but

$$CM = r; \quad Cx = d;$$

therefore

$$\frac{\cos\theta}{r} = \frac{1}{d};$$

substituting this in (7.20), we find

$$R_r = \frac{2P}{\pi d} = \text{const.} \tag{7.21}$$

Thus the stress R_r is the same at all points of the circumference concerned. Hence each circle with centre under the load P and

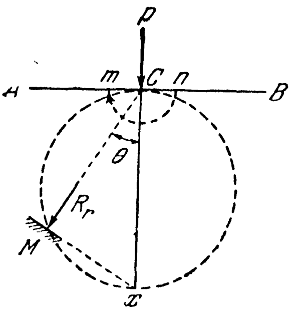

Fig. 72

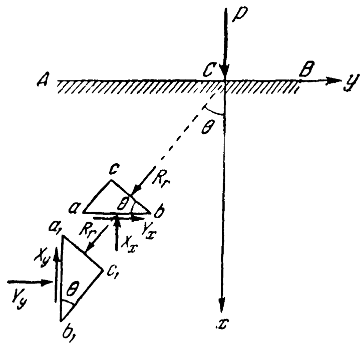

Fig. 73

passing through its point of application C is a trajectory of equal stresses R_r [1].

Let us now find the stresses in the plate on a horizontal area at a distance x from the surface. From the conditions of equilibrium of elementary prisms abc and $a_1b_1c_1$ (Fig. 73) we obtain [2]

$$\left.\begin{aligned} X_x &= R_r \cos^2\theta, \\ Y_x &= R_r \sin\theta\cos\theta, \\ Y_y &= R_r \sin^2\theta. \end{aligned}\right\} \tag{7.22}$$

Introducing hère the value of R_r from (7.20) and taking into account that

$$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \tag{7.23}$$

we have

$$X_x = -\frac{2P}{\pi}\frac{x^3}{(x^2+y^2)^2}, \qquad Y_x = -\frac{2P}{\pi}\frac{x^2y}{(x^2+y^2)^2}. \tag{7.24}$$

[1] We may recall that the points inside the semicircle mn are excluded.

[2] Cf. formulas (1.15) of Sec. 4.

Fig. 74 shows the diagrams of these stresses. In practical calculations, it is the custom to distribute the action of the load P at a given depth x under a certain angle; in the conditions of the problem considered we have

$$\max X_x = -\frac{P}{\frac{\pi x}{2}} = -\frac{P}{1.57x}.$$

Thus, for calculation one may assume conditionally that the load P is distributed uniformly over an area about $1.6x$ long which corresponds to an angle of $\theta \approx 38°$ (Fig. 75).

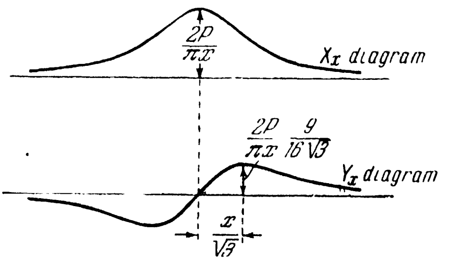

Fig. 74

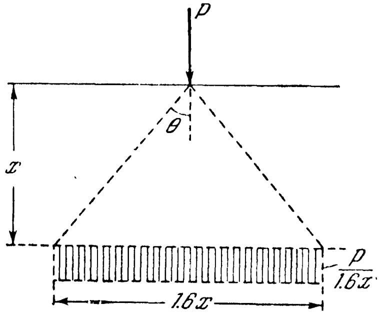

Fig. 75

If there are several loads applied at different points of the straight line AB (Fig. 73), we shall readily find stresses (7.22)

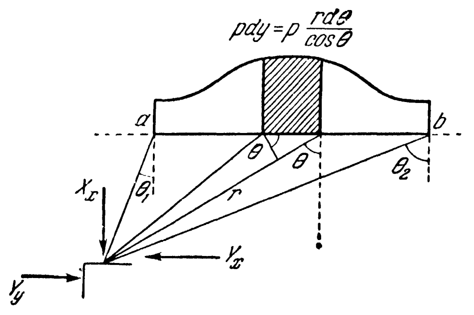

Fig. 76

at any point by summing up the effects of individual loads. Further, the problem can easily be extended to cover the case of any continuous load (Fig. 76). If the intensity of the load at a given point is p, the load acting on an infinitesimal element dy of the straight line ab will be pdy, but we see from the drawing

that

$$dy = \frac{r\,d\theta}{\cos\theta},$$

and the element of load is

$$\frac{pr\,d\theta}{\cos\theta}. \tag{7.25}$$

We substitute this quantity for the load in the first of equations (7.20) and get

$$R_r = -\frac{2p}{\pi}\,d\theta.$$

Introducing this value of R_r in equations (7.22), we obtain the stresses caused by one element of load (7.25):

$$X_x = -\frac{2p}{\pi}\cos^2\theta\,d\theta,$$

$$X_y = -\frac{2p}{\pi}\sin\theta\cos\theta\,d\theta$$

and similarly

$$Y_y = -\frac{2p}{\pi}\sin^2\theta\,d\theta.$$

The meaning of the angle θ is clear from the drawing (Fig. 76). If the load extends from point $a(\theta = \theta_1)$ to point $b(\theta = \theta_2)$, by summing up the stresses due to its individual elements, we obtain the final expressions of stresses:

$$\left.\begin{aligned} X_x &= -\frac{2}{\pi}\int_{\theta_1}^{\theta_2} p\cos^2\theta\,d\theta, \\ X_y &= -\frac{2}{\pi}\int_{\theta_1}^{\theta_2} p\sin\theta\cos\theta\,d\theta, \\ Y_y &= -\frac{2}{\pi}\int_{\theta_1}^{\theta_2} p\sin^2\theta\,d\theta. \end{aligned}\right\} \tag{7.26}$$

One sees from these formulas that it is necessary first of all to represent the load p as a function of the angle θ. If the load is uniform, p is constant; putting it before the integral signs in equations (7.26), we readily evaluate the latter and obtain

$$\left.\begin{aligned} X_x &= -\frac{p}{2\pi}\left[2\theta + \sin 2\theta\right]_{\theta_1}^{\theta_2}, \\ Y_y &= -\frac{p}{2\pi}\left[2\theta - \sin 2\theta\right]_{\theta_1}^{\theta_2}, \\ X_y = Y_x &= \frac{p}{2\pi}\cos 2\theta\,\Big|_{\theta_1}^{\theta_2}. \end{aligned}\right\} \tag{7.27}$$

These formulas can be transformed to the Cartesian co-ordinates x and y, taking into account equations (7.23) and adding to them

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \operatorname{arc}\tan\frac{y}{x}.$$

The problems concerning the stresses in a medium bounded by a plane are gaining in importance nowadays in the theory of bases and foundations for the kinds of soil which maintain, under moderate pressures, the properties close to those of a homogeneous elastic body.

50. WEDGE LOADED AT THE VERTEX

Solution (7.15) can be applied to a more general problem, introducing in it an additional arbitrary parameter by replacing θ by $(\theta-\theta_0)$ where θ_0 is so far an arbitrary angle. We obtain then

$$R_r = -k\frac{\cos(\theta-\theta_0)}{r} = -k\cos\theta_0\frac{\cos\theta}{r} - k\sin\theta_0\frac{\sin\theta}{r}. \quad (7.28)$$

The first term in the right-hand member, as we already know, satisfies all basic equations of the plane problem. The second

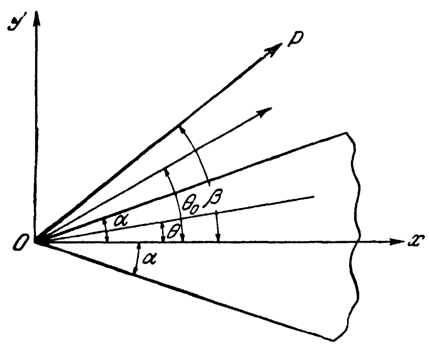

Fig. 77

term satisfies them too, since it is obtainable from the first term by rotating the polar axis through 90°, i.e., by replacing θ by $\left(\frac{\pi}{2}-\theta\right)$.

Consider (Fig. 77) a wedge to the vertex of which an arbitrarily directed force P is applied. Let the polar axis Ox be directed along the axis of the wedge. The constants k and θ_0 (7.28)

are chosen, just as in Sec. 49, from the condition that the normal forces

$$R_r \rho \, d\theta$$

along the arc mn of any radius ρ are equivalent to the given force P. Projecting all forces on the axes Ox and Oy, we get

$$\int_{-\alpha}^{+\alpha} R_{-r} \rho \, d\theta \cos\theta = P\cos\beta;$$

$$\int_{-\alpha}^{+\alpha} R_{-r} \rho \, d\theta \sin\theta = P\sin\beta.$$

Substituting here the value $R_{-r} = -R_r$ from (7.28), we obtain the following equations for the determination of k and θ_0:

$$k\cos\theta_0 \int_{-\alpha}^{+\alpha} \cos^2\theta \, d\theta + k\sin\theta_0 \int_{-\alpha}^{+\alpha} \sin\theta\cos\theta \, d\theta = P\cos\beta,$$

$$k\cos\theta_0 \int_{-\alpha}^{+\alpha} \cos\theta\sin\theta \, d\theta + k\sin\theta_0 \int_{-\alpha}^{+\alpha} \sin^2\theta \, d\theta = P\sin\beta.$$

Evaluating the integrals entering here, we find

$$\left.\begin{aligned} &\int_{-\alpha}^{+\alpha} \sin\theta\cos\theta \, d\theta = 0, \\ A = &\int_{-\alpha}^{+\alpha} \cos^2\theta \, d\theta = \frac{1}{2}(2\alpha + \sin 2\alpha), \\ B = &\int_{-\alpha}^{+\alpha} \sin^2\theta \, d\theta = \frac{1}{2}(2\alpha - \sin 2\alpha). \end{aligned}\right\} \tag{7.29}$$

Introducing this in the preceding equations, we obtain

$$Ak\cos\theta_0 = P\cos\beta; \quad Bk\sin\theta_0 = P\sin\beta.$$

Hence we readily find that

$$\left.\begin{aligned} k &= P\sqrt{\frac{\cos^2\beta}{A^2} + \frac{\sin^2\beta}{B^2}} \\ \tan\theta_0 &= \frac{A}{B}\tan\beta. \end{aligned}\right\} \tag{7.30}$$

Examining formula (7.28), we observe that it has the same form as formula (7.15), but the polar angle in it $(\theta - \theta_0)$ is

measured from a certain direction θ_0 (Fig. 77), defined by the second formula of (7.30); since in general

$$\frac{A}{B} \neq 1,$$

then $\theta_0 \neq \beta$, and this direction is not coincident in the general case with the direction of the load P. If we put $\alpha = \frac{\pi}{2}$, we shall return to the "half-plane" considered in Sec. 49, but loaded by

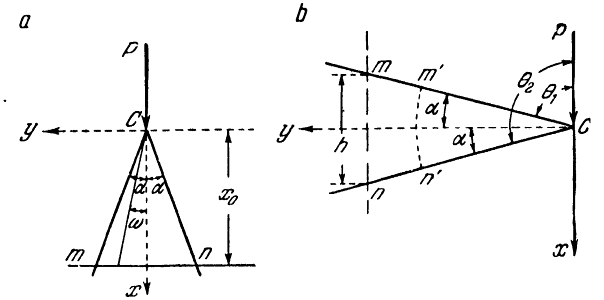

Fig 78

an inclined force (at the angle β to the normal to the plane); then from (7.29) and (7.30)

$$A = B = \frac{\pi}{2}; \quad k = \frac{2P}{\pi}; \quad \tan\theta_0 = \tan\beta; \quad \theta_0 = \beta.$$

Hence

$$R_r = -\frac{2P}{\pi}\frac{\cos(\theta - \beta)}{r},$$
$$R_\theta = \Theta_r = 0.$$

Here $(\theta - \beta)$ is (Fig. 77) the polar angle measured from the direction of the load P. Thus, solution (7.20) can immediately be applied to the case when the load P (Fig. 72) acts not normally to the plane AB but at any angle; one chould only remember that the polar angle θ is always measured from the direction of the load P. Consequently, this solution can be employed in the case of a load applied in the direction AB; the previous solution (Sec. 49) also applies to the problem of a continuous tangential load over the plane AB. Combining the cases of normal and tangential loadings, we can solve the problem involving any inclined continuous load.

As illustrative examples of the analysis of a wedge consider two cases: compression of a wedge (Fig. 78*a*) and bending of a wedge by a force applied at the end (Fig. 78*b*).

In the first case

$$\beta = 0$$

and from equations (7.29) and (7.30) we have

$$\theta_0 = 0; \quad k = \frac{2P}{2\alpha + \sin 2\alpha}.$$

These values of k and θ_0 must be introduced in formula (7.28).

Similarly to (7.22), one can find stresses over the section of the wedge mn (Fig. 78a) at a depth x_0; the normal stress is

$$X_x = -\frac{kx_0^3}{(x_0^2 + y^2)^2}.$$

If ω (Fig. 78a) denotes the angle of inclination of the radius-vector of a point (x_0, y) in the section mn, the preceding formula can be written down as

$$X_x = -\frac{k}{x_0} \frac{1}{(1 + \tan^2 \omega)^2}, \tag{7.31}$$

where

$$-\alpha \leqslant \omega \leqslant \alpha.$$

In the case of an acute wedge (angle α is small) the stress X_x deviates slightly from the value

$$X_x = -\frac{k}{x_0} = -\frac{2P}{(2\alpha + \sin 2\alpha)\, x_0},$$

i.e., the stress distribution over the section mn is nearly uniform. If, however, the wedge is obtuse (angle α is not small), the stresses are distributed highly nonuniformly; for instance, at $\alpha = 45°$ X_x varies

$$\text{from } -\frac{k}{x_0} \text{ to } -\frac{k}{4x_0}.$$

In the case of bending (Fig. 78b) we have

$$\beta = \frac{\pi}{2}$$

and from equations (7.29) and (7.30)

$$\theta_0 = \frac{\pi}{2}; \quad k = \frac{2P}{2\alpha - \sin 2\alpha}.$$

From (7.28) we have

$$R_r = -k \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{r} = -k \frac{\sin \theta}{r}.$$

But $\left(\frac{\pi}{2} - \theta\right)$ is the polar angle measured from the direction of the load; consequently, solution (7.15), where k has the above

value, holds true in this case as well. The stresses over the section mn, as previously, will be found similarly to (7.22) by the use of equations (7.15) but bearing in mind that the angle θ_1, as we have just said, should be measured from the direction of the load as shown in Fig. 78*b*. The stresses over the cross section mn in this case are expressed by

$$\left.\begin{aligned} Y_y &= -\frac{2Mx\sin^4\theta}{(2\alpha-\sin 2\alpha)\,y^3}, \\ X_y &= -\frac{2Px^2\sin^4\theta}{(2\alpha-\sin 2\alpha)\,y^3}, \end{aligned}\right\} \tag{7.32}$$

where $M=Py$ is the magnitude of the bending moment at the section.

If the angle of the wedge 2α is small, then expanding $\sin 2\alpha$ in a series, we obtain approximately

$$2\alpha-\sin 2\alpha=\frac{(2\alpha)^3}{6}.$$

After some transformations the formulas of stresses can be represented as

$$\left.\begin{aligned} Y_y &= -\frac{Mx}{J}\left(\frac{\tan\alpha}{\alpha}\right)^3\sin^4\theta, \\ X_y &= -\frac{12Px^2}{h^3}\left(\frac{\tan\alpha}{\alpha}\right)^3\sin^4\theta, \end{aligned}\right\} \tag{7.33}$$

where h is the height of the section mn, J is its moment of inertia. For small α the quantity $\frac{\tan\alpha}{\alpha}$ is close to unity; furthermore, since

$$\frac{\pi}{2}-\alpha\leqslant\theta\leqslant\frac{\pi}{2}+\alpha,$$

then for small α the angle θ is close to $\frac{\pi}{2}$ everywhere and $\sin\theta\cong 1$. It is easy to see, therefore, that the magnitude of the normal stress Y_y in equation (7.33) is close to that given by the elementary solution. The distribution of shearing stresses differs essentially from the elementary result according to the formula

$$X_y=\frac{QS}{Jb};$$

one sees from equations (7.33) that they vanish on the axis of the wedge (at $x=0$) and reach their maximum value at the edges of the section. If, however, in place of the plane section mn we make a section of the rod by a cylindrical surface shown in the drawing by the arc $m'n'$ with centre at C, then there will apparently be no shearing stresses in this section at all.

51. GENERAL SOLUTION OF THE PLANE PROBLEM IN POLAR CO-ORDINATES

Let us seek particular solutions for the basic differential equation (VIIIp) of the plane problem by using the method of separation of variables applied to equation (IX) in Sec. 44:

$$\nabla^2 (\nabla^2 \varphi) = 0, \tag{7.34}$$

where for the polar co-ordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}; \tag{7.35}$$

its solutions will be sought in the form

$$\varphi = rR\Theta, \tag{7.36}$$

where R is a function of one variable r, and Θ is a function of one variable θ. The factor r in (7.36) has been introduced for convenience in carrying out further computations.

For substitution in (7.35) and (7.34) we calculate

$$\frac{1}{r}\frac{\partial \varphi}{\partial r} = \left(\frac{R}{r} + R'\right)\Theta,$$

$$\frac{\partial^2 \varphi}{\partial r^2} = (2R' + rR'')\Theta,$$

$$\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = \frac{1}{r} R\Theta''.$$

Adding up these equalities by members, we obtain

$$\nabla^2 \varphi = \frac{R}{r}\Theta'' + \left(rR'' + 3R' + \frac{R}{r}\right)\Theta, \tag{7.37}$$

or

$$\nabla^2 \varphi = \psi_1 + \psi_2,$$

where

$$\psi_1 = r\frac{R}{r^2}\Theta'', \tag{7.38}$$

$$\psi_2 = r\left(R'' + 3\frac{R'}{r} + \frac{R}{r^2}\right)\Theta. \tag{7.39}$$

Hence

$$\nabla^2 (\nabla^2 \varphi) = \nabla^2 \psi_1 + \nabla^2 \psi_2. \tag{7.40}$$

But the expressions of ψ_1 and ψ_2 have the same structure as φ in (7.36); therefore, we shall calculate

$$\nabla^2 \psi_1 \quad \text{and} \quad \nabla^2 \psi_2$$

according to the former formula (7.37), if when calculating $\nabla^2\psi_1$ we replace R by $\frac{R}{r^2}$ and Θ by Θ'', and when calculating $\nabla^2\psi_2$ we replace R by $R''+3\frac{R'}{r}+\frac{R}{r^2}$ leaving Θ unaltered.

In this way we obtain

$$\nabla^2\psi_1=\frac{R}{r^3}\Theta^{\mathrm{IV}}+\left[r\left(\frac{R}{r^2}\right)''+3\left(\frac{R}{r^2}\right)''+\frac{R}{r^3}\right]\Theta'',$$

$$\nabla^2\psi_2=\left(\frac{R''}{r}+3\frac{R'}{r^2}+\frac{R}{r^3}\right)\Theta''+\left[r\left(R''+3\frac{R'}{r}+\frac{R}{r^2}\right)''+\right.$$
$$\left.+3\left(R''+3\frac{R'}{r}+\frac{R}{r^2}\right)'+\frac{1}{r}\left(R''+3\frac{R'}{r}+\frac{R}{r^2}\right)\right]\Theta.$$

It remains to differentiate the expressions

$$\frac{R}{r^2}\quad\text{and}\quad\left(R''+3\frac{R'}{r}+\frac{R}{r^2}\right),$$

as indicated by primes in brackets and substitute the results in (7.40). Having performed these computations, we write down equation (7.34) in the following form (multiplying both of its members by r^3):

$$R\Theta^{\mathrm{IV}}+2A\Theta''+B\Theta=0, \tag{7.41}$$

where the following notations are introduced:

$$\left.\begin{aligned}A&=R''r^2+R'r+R,\\ B&=R^{\mathrm{IV}}r^4+6R'''r^3+5R''r^2-R'r+R.\end{aligned}\right\} \tag{7.42}$$

In equation (7.41) the variables can easily be separated; dividing both of its members by R, we find

$$\Theta^{\mathrm{IV}}+2\frac{A}{R}\Theta''+\frac{B}{R}\Theta=0; \tag{7.43}$$

by partial differentiation with respect to r we obtain

$$2\left(\frac{A}{R}\right)'\Theta''+\left(\frac{B}{R}\right)'\Theta=0. \tag{7.44}$$

First consider the case when

$$\left(\frac{A}{R}\right)'\neq 0. \tag{7.45}$$

In this case, dividing both members of (7.44) by $2\left(\frac{A}{R}\right)'\Theta$, we get

$$\frac{\Theta''}{\Theta}=-\frac{\left(\frac{B}{R}\right)'}{2\left(\frac{A}{R}\right)'}=-m^2;$$

here m is a constant number. The variables are separated:

$$\Theta'' + m^2\Theta = 0; \tag{7.46}$$

$$\left(\frac{B}{R}\right)' - 2m^2\left(\frac{A}{R}\right)' = 0. \tag{7.46a}$$

The general solution of (7.46) is known:

$$\Theta = D_{m1}\cos m\theta + D_{m2}\sin m\theta. \tag{7.47}$$

Further we have

$$\Theta'' = -m^2\Theta,$$

$$\Theta^{\text{IV}} = -m^2\Theta'' = m^4\Theta.$$

Substituting this in (7.41), we have

$$Rm^4 - 2Am^2 + B = 0$$

or, writing out this by the use of (7.42), we obtain

$$R^{\text{IV}}r^4 + 6R'''r^3 + (5 - 2m^2)R^2r^2 - \\ - (1 + 2m^2)R'r + (1 - m^2)^2R = 0. \tag{7.48}$$

The solutions of equation (7.48) are sought in the form

$$R = r^n. \tag{7.49}$$

Introducing this in (7.48) and dividing it through by r^n, we obtain after simplification the following characteristic equation:

$$n^4 - 2(m^2 + 1)n^2 + (m^2 - 1)^2 = 0.$$

Its four roots are

$$n_1 = m + 1; \quad n_2 = m - 1; \quad n_3 = -m + 1; \quad n_4 = -m - 1. \tag{7.50}$$

Let m take on positive integral values; if $m \neq 0$ and $m \neq 1$, then there are no multiple roots among (7.50), and we obtain from (7.49) four linearly independent particular solutions of equation (7.48); then for $m=2, 3, 4, \ldots$ we obtain the following general solution of this equation:

$$R = C_{m1}r^{m+1} + C_{m2}r^{m-1} + C_{m3}r^{-m+1} + C_{m4}r^{-m-1},$$

where C_{mk} are arbitrary constants. Introducing this and (7.47) in (7.36) and summing up with respect to m, we get

$$\varphi = \sum_{m=2}^{n}(C_{m1}r^{m+2} + C_{m2}r^m + C_{m3}r^{-m+2} + \\ + C_{m4}r^{-m})(D_{m1}\cos m\theta + D_{m2}\sin m\theta). \tag{7.51}$$

Now consider the cases when (1) $m=0$, (2) $m=1$. In the first case equation (7.46) is replaced by $\Theta''=0$, whence

$$\Theta = D_{01}\theta + D_{02}.$$

The roots of (7.50) are multiple: $n_1=n_3=1$; $n_2=n_4=-1$; they give four independent solutions of equation (7.48):

$$r; \quad r\ln r; \quad \frac{1}{r}; \quad \frac{1}{r}\ln r.$$

Its general solution is

$$R = C_{01}r + C_{02}r\ln r + C_{03}\frac{1}{r} + C_{04}\frac{1}{r}\ln r.$$

According to (7.36) the function φ is

$$\varphi_0 = (C_{01}r^2 + C_{02}r^2\ln r + C_{03} + C_{04}\ln r)(D_{01}\theta + D_{02}). \qquad (7.52)$$

In the second case ($m=1$) the general solution (7.47) of equation (7.46) has the form

$$\Theta = D_{11}\cos\theta + D_{12}\sin\theta$$

and roots (7.50) become: $n_1=2$; $n_4=-2$; $n_2=n_3=0$. There are four independent solutions of equation (7.48) corresponding to them:

$$r^2; \quad \frac{1}{r^2}; \quad 1; \quad \ln r.$$

The general solution of equation (7.48) is now

$$R = C_{11}r^2 + C_{12}\frac{1}{r^2} + C_{13} + C_{14}\ln r.$$

The function φ_1, according to (7.36), is

$$\varphi_1 = \left(C_{11}r^3 + C_{12}\frac{1}{r} + C_{13}r + C_{14}r\ln r\right)(D_{11}\cos\theta + D_{12}\sin\theta). \qquad (7.53)$$

One may put here, however, $C_{13}=0$ since $r\cos\theta=x$, $r\sin\theta=y$; consequently, there results a function of the first degree in Cartesian co-ordinates.

Thus if, according to (7.45),

$$\left(\frac{A}{R}\right)' \neq 0,$$

this leads to three types of solution: (7.51), (7.52), (7.53).

Let us now turn to the case

$$\left(\frac{A}{R}\right)' = 0 \quad \text{or} \quad A = c_1R, \qquad (7.54)$$

where c_1 is an arbitrary constant. Equation (7.44) takes the form

$$\left(\frac{B}{R}\right)'\Theta = 0,$$

whence either $\left(\frac{B}{R}\right)' = 0$, i.e.,

$$B = c_2 R, \tag{7.55}$$

or $\Theta = 0$; but this trivial case is of no interest and we omit it because, according to (7.36), it leads to the solution

$$\varphi \equiv 0.$$

On the basis of (7.54) and (7.55) the basic equation (7.41) reduces to

$$\Theta^{\mathrm{IV}} + 2c_1\Theta'' + c_2\Theta = 0. \tag{7.56}$$

We add here (7.54) and (7.55):

$$A - c_1 R = 0; \quad B - c_2 R = 0,$$

and, writing them out with the use of (7.42), we obtain

$$\left.\begin{aligned} R''r^2 + R'r + (1 - c_1)R = 0, \\ R^{\mathrm{IV}}r^4 + 6R'''r^3 + 5R''r^2 - R'r + (1 - c_2)R = 0. \end{aligned}\right\} \tag{7.57}$$

Here c_1 and c_2 must be chosen so that equations (7.57) will have solutions in common, and later use can be made, of course, only of these common solutions; let us seek them in the form

$$R = r^m. \tag{7.58}$$

Substituting this in (7.57), we obtain two characteristic equations:

$$m^2 + 1 = c_1; \quad (m^2 - 1)^2 = c_2.$$

Introducing these values of c_1 and c_2 in (7.56) and (7.57), we reduce them to the form

$$\Theta^{\mathrm{IV}} + 2(m^2 + 1)\Theta'' + (m^2 - 1)^2\Theta = 0, \tag{7.59}$$

$$\left.\begin{aligned} R''r^2 + R'r - m^2 R = 0, \\ R^{\mathrm{IV}}r^4 + 6R'''r^3 + 5R''r^2 - R'r + [1 - (m^2 - 1)^2]R = 0. \end{aligned}\right\} \tag{7.60}$$

The solutions

$$r^m \quad \text{and} \quad r^{-m} \tag{7.61}$$

will satisfy both equations (7.60) (since the latter are not altered when m is replaced by $-m$). The solution of equation (7.59) is sought in the form $\Theta = e^{k\theta}$. The characteristic equation is

$$k^4 + 2(m^2 + 1)k^2 + (m^2 - 1)^2 = 0,$$

whence

$$k^2 = -(m^2 + 1) \pm 2m = -(m \pm 1)^2 = (m \pm 1)^2 i^2,$$

i.e.,

$$k = \pm (m \pm 1)\, i. \tag{7.62}$$

If

$$m \neq 0; \quad m \neq 1, \tag{7.63}$$

then (7.62) gives four different roots and (7.59) has four particular solutions of the form

$$\cos (m+1)\,\theta; \ \sin (m+1)\,\theta; \ \cos (m-1)\,\theta; \ \sin (m-1)\,\theta. \tag{7.64}$$

Combining them with (7.61), we observe that we obtain, according to (7.36), the values of the function φ coinciding with those of which the previously obtained solution (7.51) is composed; consequently, in the case (7.63) our investigation will give no new solutions.

It remains to investigate the cases: (1) $m=0$, (2) $m=1$. In examining these cases, we shall seek again, for greater clearness, the solutions of equations (7.60) in the form

$$R = r^n.$$

We obtain the following characteristic equations:

$$\left.\begin{aligned} n^2 &= m^2, \\ (n^2-1)^2 &= (m^2-1)^2; \end{aligned}\right\} \tag{7.65}$$

they have a common solution: $n_1 = m$; $n_2 = -m$. For $m=0$, $n_1 = n_2 = 0$ we have two particular solutions:

$$r^0 = 1; \quad r^0 \ln r = \ln r.$$

Consequently,

$$R = \bar{C}_{01} + \bar{C}_{02} \ln r. \tag{7.66}$$

Equation (7.59) will be written down as

$$\Theta^{\mathrm{IV}} + 2\Theta'' + \Theta = 0.$$

Its characteristic equation $k^4 + 2k^2 + 1 = 0$ has two double roots: $k_1 = i$; $k_2 = -i$ which give four particular solutions:

$$\cos\theta; \quad \sin\theta; \quad \theta\cos\theta; \quad \theta\sin\theta. \tag{7.67}$$

By using (7.66) and (7.67), we construct the stress function

$$\varphi = (\bar{D}_{01}\cos\theta + \bar{D}_{02}\sin\theta + \bar{D}_{03}\theta\cos\theta + \bar{D}_{04}\theta\sin\theta)\,(\bar{C}_{01}r + \bar{C}_{02}r\ln r).$$

In comparison with (7.53) there appear new solutions here:

$$\bar{\varphi}_0 = (\bar{C}_{01}r + \bar{C}_{02}r\ln r)\,(\bar{D}_{03}\theta\cos\theta + \bar{D}_{04}\theta\sin\theta). \tag{7.68}$$

We proceed to the second case $m=1$. Equations (7.65) have the roots $n_1 = 1$, $n_2 = -1$ which give two solutions:

$$r; \quad \frac{1}{r}. \tag{7.69}$$

Equation (7.59) will be written down as

$$\Theta^{\mathrm{IV}}+4\Theta''=0.$$

Its general solution is

$$\Theta=\bar{\bar{D}}_{11}\cos 2\theta+\bar{\bar{D}}_{12}\sin 2\theta+\bar{\bar{D}}_{13}\theta+\bar{\bar{D}}_{14}. \tag{7.70}$$

Expressions (7.69) and (7.70) give the following stress function:

$$\varphi=(\bar{\bar{C}}_{11}r^2+\bar{\bar{C}}_{13})(\bar{\bar{D}}_{11}\cos 2\theta+\bar{\bar{D}}_{12}\sin 2\theta+\bar{\bar{D}}_{13}\theta+\bar{\bar{D}}_{14}).$$

Comparing this with (7.51) and (7.52), we notice that we obtain no new solutions here.

Summarising the results of the integration of the basic differential equation of the plane problem (7.34), we obtain the final value of the stress function as the sum of solutions (7.51), (7.52), (7.53), (7.68). It is obvious that one can set all coefficients D equal to unity in formula (7.51) and others listed above without loss in generality; each term of the sum (7.51) can be replaced by two terms collecting the coefficients of $\cos m\theta$ and $\sin m\theta$; the constants C_{mk} $(k=1, 2, 3, 4)$ may be taken different in the two cases; the same applies to formulas (7.52), (7.53) and (7.68).

In this way we obtain

$$\varphi={}_1\varphi_0+{}_2\varphi_0+{}_1\varphi_1+{}_2\varphi_1+{}_1\bar{\varphi}_0+{}_2\bar{\varphi}_0+{}_1\varphi_m+{}_2\varphi_m, \tag{7.71}$$

where

$$\left.\begin{aligned}
{}_1\varphi_0 &= C_{01}r^2+C_{02}r^2\ln r+C_{03}+C_{04}\ln r,\\
{}_2\varphi_0 &= (C'_{01}r^2+C'_{02}r^2\ln r+C'_{03}+C'_{04}\ln r)\,\theta,\\
{}_1\varphi_1 &= \left(C_{11}r^3+C_{12}\frac{1}{r}+C_{13}r+C_{14}r\ln r\right)\cos\theta,\\
{}_2\varphi_1 &= \left(C'_{11}r^3+C'_{12}\frac{1}{r}+C'_{13}r+C'_{14}r\ln r\right)\sin\theta,\\
{}_1\bar{\varphi}_0 &= (\bar{C}_{01}r+\bar{C}_{03}r\ln r)\,\theta\cos\theta,\\
{}_2\bar{\varphi}_0 &= (\bar{\bar{C}}_{01}r+\bar{\bar{C}}_{02}r\ln r)\,\theta\sin\theta,\\
{}_1\varphi_m &= \sum_{m=2}^{n}(C_{m1}r^{m+2}+C_{m2}r^{m}+C_{m3}r^{-m+2}+C_{m4}r^{-m})\cos m\theta,\\
{}_2\varphi_m &= \sum_{m=2}^{n}(C'_{m1}r^{m+2}+C'_{m2}r^{m}+C'_{m3}r^{-m+2}+C'_{m4}r^{-m})\sin m\theta.
\end{aligned}\right\} \tag{7.72}$$

Here one can set $C_{03}=C_{13}=C'_{13}=0$, since the expressions of the function φ corresponding to these constants yield no stresses. The expression of function (7.71) was found by J. H. Michell except

for the terms

$$C'_{02}r^2 \ln r \cdot \theta; \quad C'_{04} \ln r \cdot \theta; \quad \bar{C}_{02} r \ln r \cdot \theta \cos\theta; \quad C_{02} r \ln r \cdot \theta \sin\theta, \quad (7.73)$$

which did not enter in the expression given by him [1].

The components of the state of stress corresponding to these terms are in polar co-ordinates:

For $\varphi = C'_{04} \ln r\theta$

$$\left.\begin{aligned} R_r &= C'_{04}\frac{A\theta}{r^2},\\ \Theta_\theta &= -C'_{04}\frac{\theta}{r^2},\\ R_\theta &= C'_{04}\frac{\ln r - 1}{r^2}. \end{aligned}\right\}$$

For $\varphi = C'_{02}r^2 \ln r\theta$

$$\left.\begin{aligned} R_r &= C'_{02}(2\ln r + 1)\,\theta,\\ \Theta_\theta &= C'_{02}(2\ln r + 3)\,\theta,\\ R_\theta &= -C'_{02}(\ln r + 1). \end{aligned}\right\}$$

For $\varphi = C_{02} r \ln r \cdot \theta \sin \theta$

$$\left.\begin{aligned} R_r &= C_{02}\left(\frac{2\ln r}{r}\cos\theta + \frac{1}{r}\theta\sin\theta\right),\\ \Theta_\theta &= C_{02}\frac{1}{r}\theta\sin\theta,\\ R_\theta &= -C_{02}\left(\frac{1}{r}\sin\theta + \frac{1}{r}\theta\cos\theta\right). \end{aligned}\right\}$$

For $\varphi = \bar{C}_{02} r \ln r\theta \cos\theta$

$$\left.\begin{aligned} R_r &= \bar{C}_{02}\left(\frac{1}{r}\theta\cos\theta - \frac{2\ln r}{r}\sin\theta\right),\\ \Theta_\theta &= \bar{C}_{02}\frac{1}{r}\theta\cos\theta,\\ R_\theta &= -\bar{C}_{02}\left(\frac{1}{r}\cos\theta - \frac{1}{r}\theta\sin\theta\right). \end{aligned}\right\}$$

If the pole of a system of polar co-ordinates lies within the body under investigation, all these functions are multiple-valued and cannot be used; in a simply connected region, however, these components are single-valued, if the pole of the co-ordinate system is placed outside an elastic body or at its boundary.

From solution (7.71) in the present chapter only some of the first terms have been used. In fact, in both problems of Sec. 48

[1] The most complete list of particular solutions of equation (7.34) containing both these solutions and two more solutions of the form $\cos(m \ln r) \cosh m\theta$, $r^2 \cos(m \ln r) \cosh m\theta$ may be found in the book by C. B. Biezeno and R. Grammel, *Technische Dynamik*, Berlin, Springer, 1939.

the solution is based on the function [cf. formulas (7.5)]

$$_1\varphi_0 = C_{01}r^2 + C_{02}r^2 \ln r + C_{04} \ln r.$$

In problems of Sections 48-50 the functions $_1\bar{\varphi}_0$ and $_2\bar{\varphi}_0$ are used partly [cf. formula (7.19)]. The remaining terms of function (7.71) enable us to solve many important problems related to the analysis of a wedge and closed ring [1]. The stresses corresponding to solution (7.72) will be found according to formulas (VIIp).

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[1] See S. P. Timoshenko, *Theory of Elasticity*, New York, McGraw-Hill Book Co., 1934 and P. F. Papkovitch, *Theory of Elasticity*, Moscow, Oborongiz, 1939.

VIII

Torsion of Prismatical Bars and Bending

52. TORSION OF PRISMATICAL BARS

Consider the problem of torsion of a prismatical or cylindrical bar (with the cross section of arbitrary shape) by couples lying in the planes of its extreme sections. The influence of the own weight of the bar is neglected, i.e., we assume $X=Y=Z=0$ in equations (VII) (Sec. 25).

The problem will be solved in terms of displacements (Sec. 25) by the use of Saint-Venant's semi-inverse method, i.e., a part of displacements will be prescribed and the remaining ones will be found from Lamé's equations (VI) and surface conditions (II) or (VIa).

We take the axes of co-ordinates according to Fig. 79 and, following Saint-Venant, assume that displacements u and v have the same values as in the case of a bar of circular cross section [formulas (5.19) of Sec. 32]:

$$\left.\begin{aligned} u &= -\tau yz, \\ v &= \tau xz. \end{aligned}\right\} \tag{8.1}$$

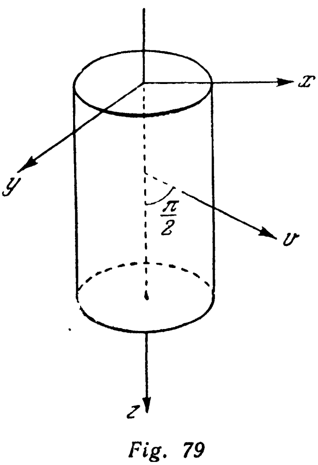

Fig. 79

The assumption $w=0$ made for the circular section has to be rejected as we introduce there by the hypothesis that the plane cross sections of a bar remain plane in torsion; in the meantime such a hypothesis in the general case of an arbitrary cross section cannot be true[1]; accordingly, we assume that

$$w = \tau\varphi(x, y), \tag{8.2}$$

[1] This will be proved at the end of the present section.

where the function φ defines the form of the warped surface of the cross section; it is called the *torsion function.* We take it to be independent of z, i.e., we assume that the warping is the same for all cross sections.

It is necessary first to check whether assumptions (8.1) and (8.2) satisfy Lamé's equations (VI) and to determine the function φ so as to satisfy the conditions on the surface of the bar corresponding to the above-stated problem of its torsion. For substitution in equations (VI) we calculate, by using (8.1) and (8.2),

$$\left.\begin{aligned} \theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0;\\ \nabla^2 u=0;\quad \nabla^2 v=0;\quad \nabla^2 w=\tau\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right). \end{aligned}\right\}\tag{8.3}$$

The first two of equations (VI) are satisfied identically while the last equation becomes

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0.\tag{8.4}$$

This is Laplace's equation in the plane; consequently, $\varphi(x, y)$ should be a harmonic function. For its determination it is necessary to add surface conditions (II) to (8.4):

$$\left.\begin{aligned} X_v&=X_x\cos(vx)+X_y\cos(vy)+X_z\cos(vz),\\ Y_v&=Y_x\cos(vx)+Y_y\cos(vy)+Y_z\cos(vz),\\ Z_v&=Z_x\cos(vx)+Z_y\cos(vy)+Z_z\cos(vz). \end{aligned}\right\}\tag{8.5}$$

By using (8.1), (8.2), (8.3), Cauchy's equations (III) and Hooke's law (V′), we calculate the components of the stress tensor:

$$\left.\begin{aligned} &X_x=\lambda\theta+2\mu\frac{\partial u}{\partial x}=0,\\ &Y_y=\lambda\theta+2\mu\frac{\partial v}{\partial y}=0,\quad X_y=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0;\\ &Z_z=\lambda\theta+2\mu\frac{\partial w}{\partial z}=0, \end{aligned}\right\}\tag{8.6}$$

$$\left.\begin{aligned} X_z&=\mu\tau\left(\frac{\partial\varphi}{\partial x}-y\right),\\ Y_z&=\mu\tau\left(\frac{\partial\varphi}{\partial y}+x\right). \end{aligned}\right\}\tag{8.7}$$

Thus on our assumptions (8.1) and (8.2) only shearing stresses act at cross sections, their distribution being the same for all sections.

Consider the boundary conditions (8.5), applying them first to the lateral cylindrical surface of the bar which is free of stresses

according to the conditions of the problem, i.e., $X_v = Y_v = Z_v = 0$. On this surface (Fig. 79) $\cos(vz) = \cos\frac{\pi}{2} = 0$. Taking into account now (8.6), we observe that the first two of conditions (8.5) are satisfied identically and the third assumes the form

$$X_z \cos(vx) + Y_z \cos(vy) = 0.$$

The meaning of this condition is very simple; if T denotes the total shearing stress at a given point, then

$$X_z = T\cos(tx); \quad Y_z = T\cos(ty),$$

and we obtain from the preceding equality

$$T[\cos(tx)\cos(vx) + \cos(tY)\cos(vy)] = 0$$

or simpler

$$T\cos(tv) = 0, \quad \text{i.e.,} \quad \cos(tv) = 0.$$

Hence the shearing stress at a point on the contour of the section is parallel to the tangent to the contour of the section at

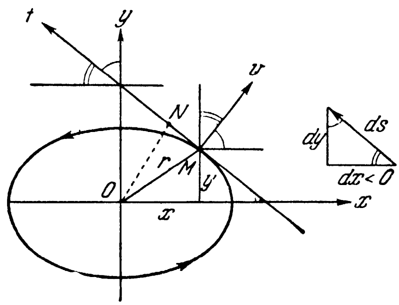

Fig. 80

this point; this condition follows directly from the law of reciprocity of shearing stresses since the lateral surface of the bar is free of forces.

Let us now express the condition on the contour of the section in terms of the torsion function φ by the use of (8.7). Multiplying by the differential element of arc of the contour ds [1], we obtain

$$\left(\frac{\partial\varphi}{\partial x} - y\right)\cos(vx)\,ds + \left(\frac{\partial\varphi}{\partial y} + x\right)\cos(vy)\,ds = 0. \qquad (8.8)$$

But from Fig. 80 we have

$$ds\cos(vx) = dy; \quad ds\cos(vy) = -dx,$$

[1] The arc is assumed to grow, i.e., $ds > 0$, when moving to the left as indicated by an arrow in Fig. 80.

and therefore from (8.8)

$$\left(\frac{\partial\varphi}{\partial x}-y\right)dy-\left(\frac{\partial\varphi}{\partial y}+x\right)dx=0,$$

or

$$\frac{\partial\varphi}{\partial x}\,dy-\frac{\partial\varphi}{\partial y}\,dx=x\,dx+y\,dy. \tag{8.9}$$

This is the boundary condition (condition on the contour of the section) which should be added to (8.4) for the determination of the harmonic function $\varphi\ (x, y)$.

The condition on the contour of the cross section (8.8) may be put in the alternate form; we write it as

$$\frac{\partial\varphi}{\partial x}\cos(vx)+\frac{\partial\varphi}{\partial y}\cos(vy)=y\cos(vx)-x\cos(vy).$$

The left-hand member represents the derivative of the function φ along the normal to the contour [cf. derivation of condition (VIa) in Sec. 25]. To transform the right-hand member we observe that from Fig. 80.

$$\cos(vx)=\cos(ty);\qquad \cos(vy)=-\cos(tx),$$

where t is the tangent to the contour of the section, furthermore,

$$x=r\cos(rx),$$

$$y=r\cos(ry),$$

where r is the radius-vector of a point M of the contour.

Therefore,

$$y\cos(vx)-x\cos(vy)=$$

$$=r[\cos(rx)\cos(tx)+\cos(ry)\cos(ty)]=r\cos(rt)=\overline{MN},$$

where $\overline{MN}$ (Fig. 80) is the projection of the radius-vector r on the tangent t. On this basis the condition on the section contour (8.8) becomes

$$\frac{\partial\varphi}{\partial v}=r\cos(rt). \tag{8.10}$$

This condition defines the value of the derivative along the normal

$$\frac{\partial\varphi}{\partial v}$$

at each point of the contour; thus, the problem of finding the torsion function $\varphi(x, y)$ is reduced to the classical Neumann problem of determining a harmonic function according to the values of its normal derivative prescribed on the contour.

The solution of the torsion problem will assume an elegant form and open new possibilities if, in addition to the harmonic function $\varphi(x, y)$, we introduce another function, the conjugate harmonic function $\psi(x, y)$ so that these two functions satisfy

the Cauchy-Riemann conditions:

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial x}&=\frac{\partial\psi}{\partial y},\\ \frac{\partial\varphi}{\partial y}&=-\frac{\partial\psi}{\partial x}.\end{aligned}\right\} \tag{8.11}$$

Introducing this in the condition on the section contour (8.9), we obtain

$$\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=x\,dx+y\,dy,$$

or

$$d\psi=d\,\frac{x^2+y^2}{2}.$$

By integrating, we get

$$\psi(x,\ y)=\frac{x^2+y^2}{2}+C, \tag{8.12}$$

where C is an arbitrary constant.

Obviously, (8.12) is the equation of that contour of the cross section for which displacements (8.1) and (8.2) will occur, i.e.,

$$u=-\tau yz;\quad v=\tau xz;\quad w=\tau\varphi(x,\ y).$$

It may be said that equation (8.12) prescribes the values of the function $\psi(x, y)$ on the contour of the section

$$\psi=\frac{r^2}{2}+C, \tag{8.13}$$

where r is the radius-vector of a point of the contour as before. Hence, another classical problem is stated for the function $\psi(x, y)$, namely the Dirichlet problem: to find a harmonic function according to the preassigned values of this function on the contour of the section.

It is now necessary to show that the obtained solution does in fact correspond to the pure torsion of a bar. It has already been proved that the lateral cylindrical surface of the bar is free of stresses. The forces applied at cross sections produce only shearing stresses (8.7); it remains to show that these forces reduce to a couple, i.e., they do not give a resultant or, in other words, their resultant force vector is equal to zero.

If the function φ in (8.7) is replaced by its conjugate function ψ according to the Cauchy-Riemann conditions (8.11), we have

$$\left.\begin{aligned}X_z&=\mu\tau\left(\frac{\partial\psi}{\partial y}-y\right),\\ Y_z&=-\mu\tau\left(\frac{\partial\psi}{\partial x}-x\right).\end{aligned}\right\} \tag{8.14}$$

We introduce a new function

$$\Phi(x, y) = \mu\tau\left(\psi - \frac{x^2+y^2}{2}\right) \tag{8.15}$$

and note that

$$\frac{\partial\Phi}{\partial x} = \mu\tau\left(\frac{\partial\psi}{\partial x} - x\right); \quad \frac{\partial\Phi}{\partial y} = \mu\tau\left(\frac{\partial\psi}{\partial y} - y\right);$$

then we obtain from (8.14) the following simple formulas for shearing stresses at a cross section

$$X_z = \frac{\partial\Phi}{\partial y}; \quad Y_z = -\frac{\partial\Phi}{\partial x}. \tag{8.16}$$

The function Φ was proposed by L. Prandtl and, by analogy with Airy's function in the plane problem, it is called the stress function. It is important to note that, according to (8.12), this function has a constant, though arbitrary, value at all points of the contour:

$$\Phi = C\mu\tau. \tag{8.17}$$

Let us calculate the projection of the resultant force vector of the shearing forces at the cross section on the axis Ox:

$$X_T = \int_F X_z\,dF = \int\int_{(F)} \frac{\partial\Phi}{\partial y}\,dx\,dy = \int_{x_0}^{x_1} dx \int_{y^0}^{y_1} \frac{\partial\Phi}{\partial y}\,dy =$$
$$= \int_{x_0}^{x_1} dx\,[\Phi(y_1) - \Phi(y_0)] = 0,$$

since $\Phi(y_1)$ and $\Phi(y_0)$—the values of the stress function at two points of the contour—are identical according to (8.17). In exactly the same way it can be shown that the other projection Y_T of the resultant force vector vanishes. This proves that the above solution of the problem corresponds to the pure torsion of the bar.

In order to complete the solution of the problem it is necessary to express the twisting moment M_t in terms of the tangential tractions at the section. We obtain

$$M_t = \int_{(F)} (Y_z x - X_z y)\,dF = -\int_{(F)} x\frac{\partial\Phi}{\partial x}\,dF - \int_{(F)} y\frac{\partial\Phi}{\partial y}\,dF. \tag{8.18}$$

Let us calculate the first of the integrals in the right-hand member

$$\int_{(F)} x\frac{\partial\Phi}{\partial x}\,dF = \int_{(F)} x\frac{\partial\Phi}{\partial x}\,dx\,dy = \int_{y_0}^{y_1} dy \int_{x_0}^{x_1} x\frac{\partial\Phi}{\partial x}\,dx.$$

Applying to the inner integral an integration by parts, we have

$$\int_{y_0}^{y_1} dy \int_{x_0}^{x_1} x \frac{\partial \Phi}{\partial x} dx = \int_{y_0}^{y_1} dy \left[x\Phi \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \Phi \, dx \right] =$$

$$= \int dy \left[x_1 \Phi(x_1, y) - x_0 \Phi(x_0, y) - \int_{x_0}^{x_1} \Phi \, dx \right]. \tag{8.19}$$

But

$$\Phi(x_1, y) \quad \text{and} \quad \Phi(x_0, y)$$

are the values of the function Φ on the contour of the section, defined by formula (8.17); the value of the constant C will not affect the magnitude of the twisting moment (8.18), since this magnitude will not be changed if Φ is replaced by $\Phi + C_1$ where C_1 is also an arbitrary constant. If the cross section is simply connected, i.e., if it is bounded by one closed contour, we may set $C=0$ in (8.17) for the sake of simplicity; then

$$\Phi(x_1, y) = \Phi(x_0, y) = 0$$

and (8.19) yields

$$\int_{y_0}^{y_1} dy \int_{x_0}^{x_1} x \frac{\partial \Phi}{\partial x} dx = - \int_{y_0}^{y_1} dy \int_{x_0}^{x_1} \Phi \, dx = - \int\int_{(F)} \Phi \, dF.$$

By similar reasoning with regard to the second integral in the right-hand member of (8.18) we find the same value for it. Thus, finally,

$$M_t = +2 \int\int_{(F)} \Phi \, dF, \tag{8.20}$$

or

$$M_t = +2\mu\tau \int\int \left(\psi - \frac{x^2 + y^2}{2} \right) dF. \tag{8.21}$$

If the twisting moment is prescribed, one can determine from this formula the angle of twist per unit length:

$$\tau = \frac{M_t}{\mu K}, \tag{8.22}$$

where

$$K = \int_{(F)} (2\psi - x^2 - y^2) \, dF \tag{8.23}$$

depends on the dimensions and shape of the section contour and represents a geometrical characteristic of the section. The quantity μK is the torsional rigidity of the bar.

From the solution here obtained one can draw an important conclusion concerning the limits of applicability of the torsion theory developed by C. A. Coulomb and based on the assumption that the plane cross sections of a bar remain plane. Indeed, in this case

$$w = \text{const} \quad \text{and} \quad \varphi(x, y) = \text{const}.$$

Then from (8.11) we find

$$\frac{\partial\psi}{\partial x} = 0; \quad \frac{\partial\psi}{\partial y} = 0;$$

hence

$$\psi(x, y) = \text{const} = C.$$

Substituting this in the equation of the section contour (8.12), we get

$$x^2 + y^2 = \text{const}.$$

Consequently, the cross section is a circle. Coulomb's theory is not applicable to any other cross sections and the cross sections will warp in torsion.

53. SAINT-VENANT'S METHOD. SPECIAL CASES

The relation, established by equations (8.11) between the torsion function $\varphi(x, y)$ and the function $\psi(x, y)$ which defines, according to (8.12), the contour of the corresponding cross section, permitted Saint-Venant to suggest the following elegant procedure of solving a number of specific problems.

Let us take an analytic function $f(z)$ of a complex variable

$$z = x + iy$$

and separate the real and imaginary parts in it:

$$f(z) = U(x, y) + iV(x, y);$$

the functions U and V will be harmonic, and they also satisfy the Cauchy-Riemann conditions:

$$\frac{dU}{dx} = \frac{dV}{dy}; \quad \frac{dU}{dy} = -\frac{dV}{dx}.$$

Let us, therefore, set $\varphi = U$; $\psi = V$. Then, if the equation

$$V(x, y) = \frac{x^2 + y^2}{2} + C$$

represents a closed curve, it will give the contour of the cross section of the bar while the function U will be the corresponding

torsion function so that the equation

$$W = \tau U(x, y)$$

will determine the displacement in the direction of the z axis, i.e., it will give the warping of the plane cross section. The stresses will be found according to formulas (8.14). Or, vice versa, one can set

$$\varphi = V(x, y); \tag{8.24}$$

then the equation of the contour (if it is closed) will be

$$-U(x, y) = \frac{x^2 + y^2}{2} + C^1 \tag{8.25}$$

and the displacement will be expressed as

$$w = \tau V(x, y). \tag{8.26}$$

Consider some special cases.

A. A Bar of Elliptic Cross Section

Let us take an analytic function

$$D_2 z^2 = D_2 (x + iy)^2 = D_2 (x^2 - y^2) + i2D_2 xy.$$

Here

$$\left.\begin{aligned} U &= D_2 (x^2 - y^2), \\ V &= 2D_2 xy. \end{aligned}\right\} \tag{8.27}$$

Assume that

$$\varphi = V = 2D_2 xy.$$

Then, according to (8.25), we obtain the equation of the contour by replacing C by $-D_1$:

$$D_2 (x^2 - y^2) + \frac{1}{2} (x^2 + y^2) = D_1,$$

or

$$\left(D_2 + \frac{1}{2}\right) x^2 + \left(\frac{1}{2} - D_2\right) y^2 = D_1,$$

or, ultimately,

$$\frac{x^2}{\left(\dfrac{D_1}{D_2 + \frac{1}{2}}\right)} + \frac{y^2}{\left(\dfrac{D_1}{\frac{1}{2} - D_2}\right)} = 1.$$

---

[1] It is taken into account here that if the function $U + iV$ is analytic, $V - iU$ is also an analytic function, since the Cauchy-Riemann conditions are identical for them.

Setting here

$$\frac{D_1}{D_2+\frac{1}{2}}=a^2, \quad \frac{D_1}{\frac{1}{2}-D_2}=b^2, \tag{8.28}$$

we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{8.29}$$

i.e., we have a bar of elliptic cross section.

From (8.28) we find the constants D_1 and D_2, expressing them in terms of the semiaxes a and b of the ellipse of the cross section:

$$D_2=\frac{1}{2}\frac{b^2-a^2}{a^2+b^2}; \quad D_1=\frac{a^2b^2}{a^2+b^2}.$$

Hence

$$w=2D_2\tau xy=\frac{b^2-a^2}{a^2+b^2}\tau xy. \tag{8.30}$$

Next, according to (8.7), we find the shearing stresses over the section by replacing the symbol μ by G:

$$\left.\begin{aligned} X_z&=G\left(\frac{b^2-a^2}{a^2+b^2}\tau y-\tau y\right)=-G\frac{2a^2}{a^2+b^2}\tau y, \\ Y_z&=G\frac{2b^2}{a^2+b^2}\tau x. \end{aligned}\right\} \tag{8.31}$$

Finally, we correlate the stresses over the section with the twisting moment M_t:

$$M_t=-\iint_F (X_z y-Y_z x)\,dF=-G\frac{2\tau}{a^2+b^2}\iint_F(-a^2y^2-b^2x^2)\,dF. \tag{8.32}$$

But

$$\iint y^2\,dF=J_x; \quad \iint x^2\,dF=J_y,$$

where J_x and J_y are the moments of inertia of the ellipse about its axes:

$$J_x=\frac{\pi ab^3}{4}; \quad J_y=\frac{\pi a^3 b}{4}.$$

Therefore, we obtain from (8.32)

$$M_t=G\frac{2\tau}{a^2+b^2}\frac{\pi a^3b^3}{2}=G\frac{\pi a^3b^3}{a^2+b^2}\tau.$$

Hence, the angle of twist τ can be expressed in terms of the twisting moment:

$$\tau=\frac{M_t}{G\dfrac{\pi a^3b^3}{a^2+b^2}}.$$

In the case of a circular section the elementary Coulomb's theory yields

$$\tau = \frac{M_t}{GJ_p},$$

where J_p is the polar moment of inertia of the section. In order to compare these expressions let us correlate the quantity

$$\frac{\pi a^3 b^3}{a^2 + b^2}$$

with the polar moment of inertia of the ellipse:

$$J_p = J_x + J_y = \frac{\pi a b^3}{4} + \frac{\pi a^3 b}{4} = \frac{\pi ab\,(a^2 + b^2)}{4} =$$

$$= \frac{\pi a^4 b^4\,(a^2 + b^2)}{4a^3 b^3} = \frac{\pi a^4 b^4}{4}\,\frac{a^2 + b^2}{a^3 b^3}\,\frac{\pi^3}{\pi^3} = \frac{F^4}{4\pi^2}\,\frac{a^2 + b^2}{\pi a^3 b^3},$$

i.e.,

$$\frac{\pi a^3 b^3}{a^2 + b^2} = \frac{F^4}{4\pi^2 J_p} \cong \frac{F^4}{39.4 J_p},$$

and, consequently,

$$\tau = \frac{M_t}{GF^4}\,39.4 J_p. \tag{8.33}$$

Replacing w by z in equality (8.30), we obtain the equation of the warped surface of the cross section $z = mxy$, where

$$m = \frac{b^2 - a^2}{a^2 + b^2}\,\tau;$$

this surface is a hyperbolic paraboloid; a set of horizontal planes $z = z_n$, where $n = 1, 2, 3, \ldots$, cuts it in hyperbolas

$$xy = \frac{z_n}{m},$$

referred to the asymptotes (Fig. 81).

By the use of equations (8.31) we can find the maximum shearing stresses. Indeed, the magnitude of the total shearing stress is given by

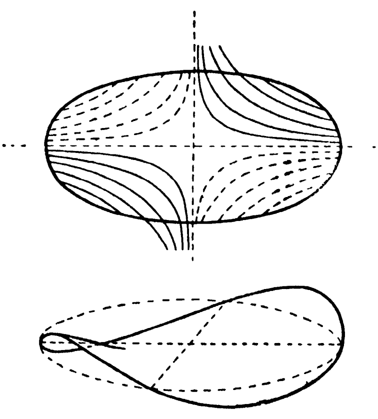

Fig. 81

$$T = \sqrt{X_z^2 + Y_z^2} = \frac{2G\tau}{a^2 + b^2}\sqrt{b^4 x^2 + a^4 y^2} =$$

$$= 2G\tau\,\frac{a^2 b^2}{a^2 + b^2}\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}. \tag{8.34}$$

Hence one can see that the stress T, as a function of x and y, remains the same

$$T = 2G\tau\,\frac{a^2 b^2}{a^2 + b^2}\,C \tag{8.35}$$

at all points of the curve

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=C^2, \tag{8.36}$$

where C is a constant number. For different values of C equation (8.36) gives a family of curves [level lines of the function $T(x, y)$]; each of these curves represents an ellipse with the semi-axes

$$a_1=a^2C; \quad b_1=b^2C. \tag{8.37}$$

If we assume that $b<a$, then

$$\frac{b_1}{a_1}=\left(\frac{b}{a}\right)^2<\frac{b}{a},$$

and the ellipse (8.36) will be more elongated than the ellipse (8.29) of the section contour. Equation (8.36) gives a family of similar and similarly located ellipses (the ratio $\frac{b_1}{a_1}=\frac{b^2}{a^2}$ is constant), the stress (8.35), which is constant for all points of the

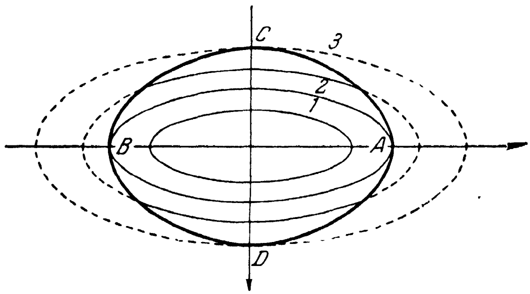

Fig. 82

ellipse, becoming greater as the dimensions of the ellipse increase (Fig. 82). It follows from this that the maximum value of the stress (corresponding to ellipse *3*) occurs at the extreme points C and D of the minor axis of the section. For these points we have from (8.37) $b_1=b=b^2C$ or $C=\frac{1}{b}$ and, therefore, according to equation (8.35)

$$T_1=\max T=2G\tau\frac{a^2b}{a^2+b^2}.$$

At the points A and B the stress (being the same at all points of the corresponding ellipse *1*) will be smaller; we have for them

$$a_1=a=a^2C; \quad C=\frac{1}{a};$$

$$T_2=2G\tau\frac{ab^2}{a^2+b^2}<T_1.$$

B. A Bar of Triangular Cross Section

Let us take an analytic function

$$C_3 z^3 = C_3 (x + iy)^3 = C_3 (x^3 - 3xy^2) + iC_3 (3x^2 y - y^3)$$

and assume

$$\varphi = V = C_3 (3x^2 y - y^3). \tag{8.38}$$

Then, according to (8.25), we obtain the equation of the contour

$$C_3 (x^3 - 3xy^2) + \frac{1}{2}(x^2 + y^2) = -C$$

or, setting

$$C_3 = \frac{1}{2} D_3; \quad -C = \frac{1}{2} D_1, \tag{8.39}$$

we have

$$D_3 (x^3 - 3xy^2) + x^2 + y^2 = D_1. \tag{8.40}$$

This equation represents a complete hyperbola of the third order with three asymptotes whose directions will be obtained in accordance with the general rule by equating to zero a set of the terms of the third degree $x^3 - 3xy^2 = x(x + y\sqrt{3})(x - y\sqrt{3}) = 0$; hence the equations of three straight lines parallel to the asymptotes are

$$x = 0; \quad x + y\sqrt{3} = 0; \quad x - y\sqrt{3} = 0.$$

Let us take three straight lines parallel to these:

$$x = a; \quad x + y\sqrt{3} = b; \quad x - y\sqrt{3} = b. \tag{8.41}$$

By a suitable choice of the constants D_1 and D_3 curve (8.40) can be shown to fall into these three straight lines, i.e., its equation will assume the form

$$D_3 (x - a)(x - b + y\sqrt{3})(x - b - y\sqrt{3}) = 0,$$

or

$$D_3 (x - a)[(x - b)^2 - 3y^2] = 0. \tag{8.42}$$

In fact, writing out equations (8.40) and (8.42), we have

$$D_3 x^3 - 3D_3 xy^2 + x^2 + y^2 - D_1 = 0, \tag{8.43}$$

$$D_3 x^3 - 3D_3 xy^2 - D_3 (a + 2b) x^2 + 3D_3 ay^2 + \\ + D_3 b (2a + b) x - D_3 ab^2 = 0. \tag{8.44}$$

By comparing the coefficients of these equations, we obtain the conditions for determining D_1, D_3 and b:

$$-D_3 (a + 2b) = 1; \quad D_3 b (2a + b) = 0;$$
$$3D_3 a = 1; \quad D_3 ab^2 = D_1.$$

Hence we find

$$b = -2a; \quad D_3 = \frac{1}{3a}; \quad D_1 = \frac{4}{3}a^2, \tag{8.45}$$

and equation (8.44) becomes

$$(x-a)[(x+2a)^2 - 3y^2] = 0$$

or

$$(x-a)(x+2a+y\sqrt{3})(x+2a-y\sqrt{3}) = 0,$$

i.e., curve (8.44) falls into three straight lines

$$x = a; \quad x+2a+y\sqrt{3} = 0; \quad x+2a-y\sqrt{3} = 0,$$

which form (Fig. 83) an equilateral triangle with centroid at the origin of co-ordinates. Thus we come to the problem of torsion of a bar of triangular cross section.

Equation (8.38), on the basis of (8.39), and (8.45), gives

$$w = \frac{\tau}{6a}(3x^2y - y^3), \tag{8.46}$$

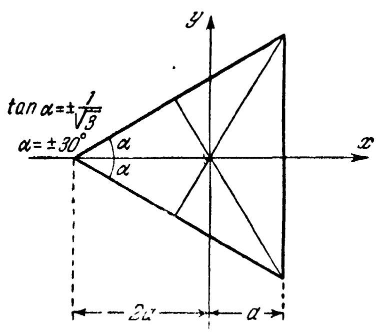

Fig. 83

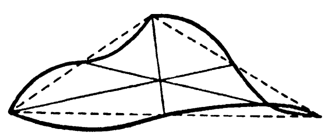

Fig. 84

i.e., the cross section warps into the surface

$$z = \frac{\tau}{6a}(3x^2y - y^3),$$

or

$$z = \frac{\tau}{6a}y(x\sqrt{3}+y)(x\sqrt{3}-y).$$

It follows that the straight lines

$$y = 0; \quad y = -x\sqrt{3}; \quad y = x\sqrt{3},$$

which are the three altitudes of the triangle of the section, remain in the original plane of the section, since $w = z = 0$ for all points of these straight lines. Fig. 84 shows the form of the warped cross section. With function (8.46) we find the stress

components X_z and Y_z according to equations (8.2) and (8.7). Further, similarly to equation (8.32), we obtain the relation between the twisting moment M_t and the angle of twist τ.

C. A Bar of Rectangular Cross Section

This problem was also solved by Saint-Venant; let us denote the lengths of the sides of the section by b and c (Fig. 85) and begin the solution of the problem with the determination of the torsion function $\varphi(x, y)$ in the form of a series. Particular solutions of equation (8.4) will be found by the method of separation of variables, which has already been applied previously, assuming that

$$\varphi = XY. \tag{8.47}$$

The variables are immediately separated:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2,$$

and hence

$$\left.\begin{aligned} X'' &= -\lambda^2 X, \\ Y'' &= \lambda^2 Y. \end{aligned}\right\} \tag{8.48}$$

As we shall see later, it is convenient to set

$$\lambda = \frac{m\pi}{b}, \quad \text{where} \quad m = 0,\ 1,\ 2,\ \ldots .$$

When $m=0$, equations (8.48) yield

$$X = \alpha x + \beta; \quad Y = \gamma x + \delta.$$

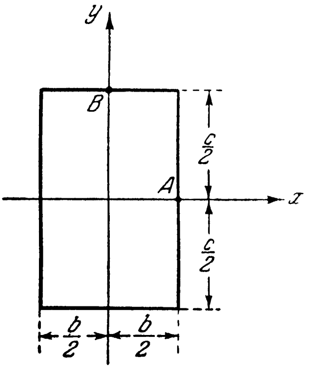

Fig. 85

Substituting this in (8.47) and retaining only the term of the second degree, we get (setting also $a = \gamma = 1$)

$$\varphi_0 = xy. \tag{8.49}$$

When $m \neq 0$ the solutions of equations (8.48) are

$$\left.\begin{aligned} X &= C_m \sin\frac{m\pi x}{b} + C_m' \cos\frac{m\pi x}{b}, \\ Y &= D_m \sinh\frac{m\pi y}{b} + D_m' \cosh\frac{m\pi y}{b}. \end{aligned}\right\} \tag{8.50}$$

By summing up solutions (8.49) and (8.50), we obtain (setting $C'_m = 0$ for all values of m)

$$\varphi = xy + \sum C_m \sin\frac{m\pi x}{b}\left(D_m \sinh\frac{m\pi y}{b} + D_m' \cosh\frac{m\pi y}{b}\right). \tag{8.51}$$

We adjust the arbitrary constants C_m, D_m and D'_m so as to satisfy the conditions on the contour; they require, as we have seen above, the shearing stresses to be directed along the tangent to the contour; consequently, on the lateral sides $x = \pm \frac{b}{2}$ (Fig. 85) we must have according to (8.7)

$$X_z = 0 \quad \text{or} \quad \frac{\partial \varphi}{\partial x} = y; \tag{8.52}$$

on the upper and lower sides $y = \pm \frac{c}{2}$

$$Y_z = 0 \quad \text{or} \quad \frac{\partial \varphi}{\partial y} = -x. \tag{8.53}$$

Turning to (8.51), we find

$$\frac{\partial \varphi}{\partial x} = y + \sum C_m \frac{m\pi}{b} \cos \frac{m\pi x}{b} \left(D_m \sinh \frac{m\pi y}{b} + D'_m \cosh \frac{m\pi y}{b} \right).$$

It follows from this that condition (8.52) will be satisfied if we assign m only odd values: $m = 1, 3, 5, \ldots$; in fact, then for $x = \pm \frac{b}{2}$ we have

$$\cos \frac{m\pi x}{b} = \cos \left(\pm \frac{m\pi}{2} \right) = 0.$$

To consider condition (8.53) we calculate from (8.51)

$$\frac{\partial \varphi}{\partial y} = x + \sum C_m \frac{m\pi}{b} \sin \frac{m\pi x}{b} \left(D_m \cosh \frac{m\pi y}{b} + D'_m \sinh \frac{m\pi y}{b} \right).$$

Introducing here the value $y = \pm \frac{c}{2}$, we write condition (8.53) in the form

$$-2x = \sum_{m=1,3,5,\ldots} C_m \frac{m\pi}{b} \left(D_m \cosh \frac{m\pi c}{2b} \pm D'_m \sinh \frac{m\pi c}{2b} \right) \sin \frac{m\pi x}{b}.$$

In view of the fact that the second term in parentheses may have two values, it is necessary to set $D'_m = 0$; in addition, we can, of course, take $D_m = 1$ to obtain our condition in the final form

$$-2x = \sum_{m=1,3,5,\ldots} \beta_m \sin \frac{m\pi x}{b}, \tag{8.54}$$

where

$$\beta_m = C_m \frac{m\pi}{b} \cosh \frac{m\pi c}{2b}. \tag{8.55}$$

It is evident from (8.54) that β_m are coefficients of the Fourier series for the function $-2x$ in the interval $\left(-\frac{b}{2}, +\frac{b}{2} \right)$; they

are determined in the usual way; multiplying both members of (8.54) by $\sin\frac{m\pi x}{b}$ and integrating, we have

$$\beta_m \frac{b}{2} = -2 \int_{-\frac{b}{2}}^{+\frac{b}{2}} x \sin\frac{m\pi x}{b}\, dx = -\frac{4b^2}{\pi^2 m^2}(-1)^{\frac{m-1}{2}} = \frac{4b^2}{\pi^2 m^2}(-1)^{\frac{m+1}{2}}\,[1],$$

$$\text{or} \quad \beta_m = \frac{8b}{\pi^2 m^2}(-1)^{\frac{m+1}{2}}.$$

On this basis we find from (8.55)

$$C_m = (-1)^{\frac{m+1}{2}} \frac{8b^2}{\pi^3 m^3 \cosh\frac{m\pi c}{2b}}.$$

Introducing this in (8.51) and taking into account that $D_m = 1$, $D'_m = 0$, we obtain the final expression of the torsion function:

$$\varphi(x, y) = xy + \frac{8b^2}{\pi^3} \sum_{m=1,3,5,\ldots} (-1)^{\frac{m+1}{2}} \frac{1}{m^3 \cosh\frac{m\pi c}{2b}} \sin\frac{m\pi x}{b} \sinh\frac{m\pi y}{b}. \tag{8.56}$$

Further, by using the Cauchy-Riemann conditions (8.11), we seek its conjugate function $\psi(x, y)$; we have

$$\frac{\partial\varphi}{\partial x} = y + \frac{8b}{\pi^2} \sum (-1)^{\frac{m+1}{2}} \frac{1}{m^2 \cosh\frac{m\pi c}{2b}} \cos\frac{m\pi x}{b} \sinh\frac{m\pi y}{b} = \frac{\partial\psi}{\partial y},$$

$$\frac{\partial\varphi}{\partial y} = x + \frac{8b}{\pi^2} \sum (-1)^{\frac{m+1}{2}} \frac{1}{m^2 \cosh\frac{m\pi c}{2b}} \sin\frac{m\pi x}{b} \cosh\frac{m\pi y}{b} = -\frac{\partial\psi}{\partial x}.$$

By integrating the first of these equations we get

$$\psi = \frac{y^2}{2} + \frac{8b^2}{\pi^3} \sum (-1)^{\frac{m+1}{2}} \frac{1}{m^3 \cosh\frac{m\pi c}{2b}} \cos\frac{m\pi x}{b} \cosh\frac{m\pi y}{b} + \omega_1(x);$$

the second yields

$$\psi = -\frac{x^2}{2} + \frac{8b^2}{\pi^3} \sum (-1)^{\frac{m+1}{2}} \frac{1}{m^3 \cosh\frac{m\pi c}{2b}} \cos\frac{m\pi x}{b} \cosh\frac{m\pi y}{b} + \omega_2(y),$$

---

[1] Here the substitution is made

$$\sin\frac{m\pi}{2} = (-1)^{\frac{m-1}{2}} \quad \text{for} \quad m = 1, 3, 5, \ldots$$

where ω_1 and ω_2 are arbitrary functions. To have both of these expressions of the function ψ identical, we must set

$$\omega_1(x) = -\frac{x^2}{2} + C,$$

$$\omega_2(y) = \frac{y^2}{2} + C,$$

C is an arbitrary constant.

We obtain finally

$$\psi = \frac{y^2 - x^2}{2} + \frac{8b^2}{\pi^3} \sum (-1)^{\frac{m+1}{2}} \frac{1}{m^3 \cosh \frac{m\pi c}{2b}} \cos \frac{m\pi x}{b} \cosh \frac{m\pi y}{b} + C. \tag{8.57}$$

We calculate now the stress function according to formula (8.15) (we take here $C = \frac{b^2}{4}$ to make this function vanish on the contour):

$$\Phi(x, y) = \mu\tau \left[\frac{b^2}{4} - x^2 + \right.$$

$$\left. + \frac{8b^2}{\pi^3} \sum_{m=1, 3, 5, \dots} (-1)^{\frac{m+1}{2}} \frac{1}{m^3 \cosh \frac{m\pi c}{2b}} \cos \frac{m\pi x}{b} \cosh \frac{m\pi y}{b} \right]. \tag{8.58}$$

To determine the angle of twist τ according to a preassigned twisting moment we use relation (8.20); carrying out integration in it we obtain [1]

$$M_t = \mu\tau \left[\frac{b^3 c}{3} + \frac{64 b^4}{\pi^5} \sum_{m=1, 3, 5, \dots} (-1)^m \frac{1}{m^5} \tanh \frac{m\pi c}{2b} \right] =$$

$$= \mu\tau b^3 c \left[\frac{1}{3} - \frac{64}{\pi^5} \frac{b}{c} \sum_{m=1, 3, 5, \dots} \frac{1}{m^5} \tanh \frac{m\pi c}{2b} \right].$$

Hence

$$\tau = \frac{M_t}{\mu b^3 c \alpha}, \tag{8.59}$$

where

$$\alpha = \frac{1}{3} - \frac{64}{\pi^5} \frac{b}{c} \sum_{m=1, 3, 5, \dots} \frac{1}{m^5} \tanh \frac{m\pi c}{2b}. \tag{8.60}$$

---

[1] In further calculations we take into account that $(-1)^m = -1$ for $m = 1, 3, 5, \dots$

We see that α depends only on the ratio of the sides of the rectangle $\frac{b}{c}$; for a square, i.e., for $\frac{b}{c}=1$, we get

$$\alpha=\frac{1}{3}-\frac{64}{\pi^5}\left(\tanh\frac{\pi}{2}+\frac{1}{243}\tanh\frac{3\pi}{2}+\ldots\right).$$

For a very narrow rectangle we assume approximately that $\frac{b}{c}=0$, and then

$$\alpha=\frac{1}{3}=0.333\ldots.$$

To complete the solution of the problem we calculate the shearing stresses by using (8.16) and (8.58):

$$X_z=\mu\tau\frac{8b}{\pi^2}\sum(-1)^{\frac{m+1}{2}}\frac{1}{m^2\cosh\frac{m\pi c}{2b}}\cos\frac{m\pi x}{b}\sinh\frac{m\pi y}{b}, \tag{8.61}$$

$$Y_z=\mu\tau\left[2x+\frac{8b}{\pi^2}\sum(-1)^{\frac{m+1}{2}}\frac{1}{m^2\cosh\frac{m\pi c}{2b}}\sin\frac{m\pi x}{b}\cosh\frac{m\pi y}{b}\right]. \tag{8.62}$$

If $b<c$, the maximum value of the shearing stress occurs at the point A (Fig. 85), i.e., for $x=\frac{b}{2}$, $y=0$; then we have from (8.62)

$$T_A=T_{\max}=\mu\tau b\left[1-\frac{8}{\pi^2}\sum\frac{1}{m^2\cosh\frac{m\pi c}{2b}}\right],$$

or by using (8.59)

$$T_{\max}=\frac{M_t}{b^2c\beta}, \tag{8.63}$$

where

$$\beta=\frac{\alpha}{1-\frac{8}{\pi^2}\sum\frac{1}{m^2\cosh\frac{m\pi c}{2b}}}.$$

The stress at the point B (Fig. 85) will be obtained from (8.61), setting $x=0$, $y=\frac{c}{2}$:

$$X_z=T_B=\mu\tau\frac{8b}{\pi^2}\sum(-1)^{\frac{m+1}{2}}\frac{1}{m^2}\tanh\frac{m\pi c}{2b},$$

or

$$T_B=\frac{M_t}{b^2c\beta_1}, \tag{8.64}$$

where

$$\beta_1 = \frac{\alpha}{\frac{8}{\pi^2}\sum(-1)^{\frac{m+1}{2}}\frac{1}{m^2}\tanh\frac{m\pi c}{2b}}.$$

The values of the coefficients α, β and β_1 introduced here are given in the table below:

$\frac{c}{b}$	1.0	1.5	1.75	2.0	2.5	3.0	4.0	6.0	8.0	10.0	∞
α	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333
β	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β_1	0.208	0.270	—	0.309	—	0.354	0.379	0.402	—	—	0.448

54. SOLUTION OF THE TORSION PROBLEM IN TERMS OF STRESSES. PRANDTL'S ANALOGY

The introduction of Prandtl's stress function makes it possible to change the whole method of solving the torsion problem and to find first the shearing stresses on the cross section (cf. Sections 17 and 36).

The torsion function $\varphi(x, y)$ and its conjugate $\psi(x, y)$ are harmonic, i.e., they satisfy Laplace's equation

$$\nabla^2\varphi = 0; \quad \nabla^2\psi = 0. \tag{8.65}$$

Let us see what differential equation the stress function $\Phi(x, y)$ satisfies. We have found above (8.14) and (8.16) that

$$\left.\begin{aligned} -Y_z &= \frac{\partial\Phi}{\partial x} = \mu\tau\left(\frac{\partial\psi}{\partial x} - x\right), \\ X_z &= \frac{\partial\Phi}{\partial y} = \mu\tau\left(\frac{\partial\psi}{\partial y} - y\right). \end{aligned}\right\} \tag{8.66}$$

We eliminate ψ by using the second of equations (8.65); differentiating the first of equations (8.66) with respect to x and the second with respect to y and adding up the results, we obtain

$$\nabla^2\Phi = -2\mu\tau. \tag{8.67}$$

Thus the stress function satisfies Poisson's differential equation (8.67). It may be seen that this equation represents the compat-

ibility equation which should be added to the equations of equilibrium, since the introduction of Prandtl's stress function allows the problem to be solved in terms of stresses. Indeed, Saint-Venant's compatibility equations are derived from Cauchy's equations (III) through the elimination of the displacements u, v, w from them. However, at the very beginning of this chapter certain restrictions were imposed on the displacements by formulas (8.1):

$$u = -\tau yz; \quad v = \tau xz,$$

and, therefore, Cauchy's equations are considerably simplified:

$$e_{xx} = e_{yy} = e_{zz} = 0; \quad e_{xy} = 0;$$

$$e_{xz} = \frac{\partial w}{\partial x} - \tau y; \quad e_{yz} = \frac{\partial w}{\partial y} + \tau x.$$

Differentiating the last two of them and subtracting the results, we eliminate the only remaining displacement w:

$$\frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y} = 2\tau.$$

Multiplying this equality by Lamé's constant μ and using Hooke's law

$$\mu e_{yz} = Y_z = -\frac{\partial \Phi}{\partial x}; \quad \mu e_{xz} = X_z = \frac{\partial \Phi}{\partial y},$$

we obtain equation (8.67). In the torsion problem it is equivalent to six equations of Beltrami (VII) of Sec. 36; just as in the case of the axisymmetrical plane problem [Sec. 48, equation (7.13)] such a simplification is a consequence of the restrictions imposed on the displacements.

It is necessary to add to equation (8.67) the previously obtained boundary condition on the section contour (8.17)

$$\Phi = \mu\tau C. \tag{8.68}$$

In the case of a simply connected cross section, i.e., a section bounded by one closed contour and, consequently, without inner cavities, the arbitrary constant C may be taken equal to zero. The determination of the stress function will then be reduced to the classical problem of integration of Poisson's equation (8.67) with the condition that the function Φ vanishes on the contour of the section:

$$\Phi = 0 \quad \text{(on the contour).} \tag{8.69}$$

If this problem is solved, the stresses on the cross section will immediately be found according to formulas (8.66) and the angle of twist τ, according to equation (8.20).

The cases considered in the preceding section can easily be solved by this method. For instance, for an elliptic section

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

the solution of equation (8.67) can be taken in the form

$$\Phi=K\mu\tau\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right), \tag{8.70}$$

which satisfies the boundary condition (8.69). Substituting this solution in (8.67), we find immediately the constant K.

This method of solution of the problem led L. Prandtl, who had proposed it, to the following analogy, which lends much clarity to the corresponding computations and at the same time makes it possible to elaborate a purely experimental method of solving the problem for any contour of the cross section of a twisted bar. Imagine a flexible inextensible membrane clamped on an elastic contour identical with the periphery of the given cross section; the tension is constant in all directions. If we apply a uniform pressure p to the membrane, it may buckle to some extent on account of small deformations of the elastic contour itself [1]; the equation of equilibrium of the membrane was derived by P. S. Laplace; it coincides with the equation given in all courses on strength of materials for the analysis of thin vessels having the form of bodies of revolution:

$$\frac{\sigma_1}{\rho_1}+\frac{\sigma_2}{\rho_2}=\frac{p}{h},$$

where σ_1 and σ_2 are the principal stresses in the wall of the vessel, ρ_1 and ρ_2 are the principal radii of curvature of the wall, and h is the wall thickness.

Taking into account that the tension of the membrane is constant, i.e., $\sigma_1=\sigma_2$ and that $h\sigma_1=h\sigma_2=H$ is the magnitude of this tension per unit length of the section of the membrane, we obtain

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{p}{H};$$

if the buckling of the membrane is small, we may assume

$$\frac{1}{\rho_1}=\frac{\partial^2 z}{\partial x^2}; \qquad \frac{1}{\rho_2}=\frac{\partial^2 z}{\partial y^2},$$

where z is the ordinate of the surface of the membrane, and we obtain the following differential equation of the surface of the

[1] One can see, however, that these deformations will be small as compared to the sag of the membrane z and they can be neglected.

membrane:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{p}{H}, \quad \text{or} \quad \nabla^2 z=\frac{p}{H}. \tag{8.71}$$

We add to it the obvious boundary condition

$$z=0 \quad \text{(on the contour).} \tag{8.72}$$

Comparing (8.71) and (8.72) with (8.67) and (8.69) respectively, we come to Prandtl's analogy: if the tension of the membrane H and the load p are chosen so that

$$\frac{p}{H}=-2\mu\tau, \tag{8.73}$$

then

$$z=\Phi; \tag{8.74}$$

hence, the ordinate of the surface of the buckled membrane gives the value of the stress function at a given point of the cross section, and the twisting moment is equal, according to (8.20), to

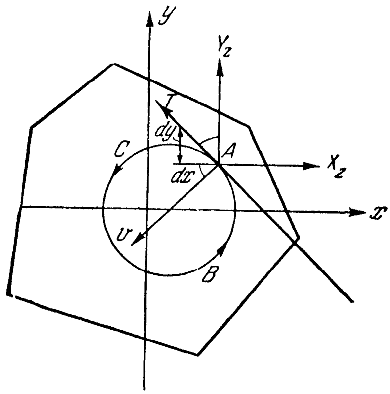

Fig. 86

double the volume bounded by the surface of the membrane and its original plane.

The distribution of the shearing stresses over the section can also be represented visually with the aid of Prandtl's analogy. Let any closed contour BC be passed through a chosen arbitrary point A of the section (Fig. 86) and a tangent AT and an inward normal Av to the contour be drawn at A. The projection of the total stress t at this point on the tangent AT will be

$$t^{(T)}=X_z\cos(xT)+Y_z\cos(yT);$$

but

$$\cos(xT) = +\cos(yv),$$
$$\cos(yT) = -\cos(xv).$$

Using also formulas (8.16), we get

$$t^{(T)} = \frac{\partial \Phi}{\partial x}\cos(xv) + \frac{\partial \Phi}{\partial y}\cos(yv),$$

or

$$t^{(T)} = \frac{\partial \Phi}{\partial v}; \tag{8.75}$$

this is a generalisation of the property of the stress function as expressed by formulas (8.16); the projection of the total stress

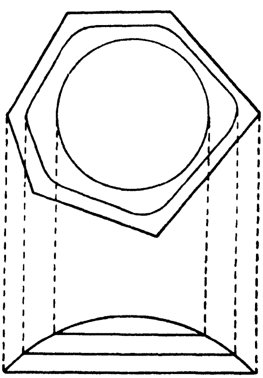

Fig. 87

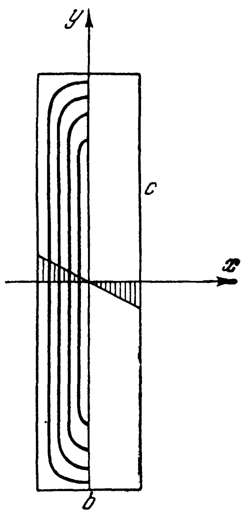

Fig. 88

on any direction T is equal to the derivative of the function Φ along the normal v to this direction.

Now let us consider the surface of the buckled membrane and imagine it has been cut by a number of equidistant planes parallel to the contour (Fig. 87). We obtain a series of closed contours—horizontals of the surface. At any point A of the horizontal the derivative $\partial\Phi/\partial T$ along the tangent to it is equal to zero since $\Phi=$const in this direction. Hence we conclude, according to

(8.75), that the projection of the total shearing stress t on the normal $A\nu$ to the horizontal is zero; consequently, the stress t is directed along the tangent to the horizontal at this point. The same formula (8.75) shows that the magnitude of the total shearing stress is equal to the derivative of the function Φ along the inward normal to the horizontal or, which is the same, to the gradient of the function Φ at a given point:

$$t = \frac{\partial \Phi}{\partial \nu} = \operatorname{grad} \Phi. \tag{8.76}$$

Taking into account that the gradient (the tangent of the maximum angle of inclination of the surface of the membrane at a given point to its initial plane) is proportional to the closeness of horizontals, we see that the system of horizontals gives a visual picture of stress distribution over the section of a twisted rod. Hence, one can draw, for instance, the following rather general conclusions.

(1) If the section is bounded by a simple contour which is everywhere convex, the closeness of horizontals increases as the contour is approached, and it is there that one should expect the greatest stresses.

(2) If the contour indicated in (1) above has two axes of symmetry and is somewhat elongated in the direction of one of these axes, the closeness of horizontals will be maximal at the ends of the minor axis as, for instance, in the case of an ellipse or a rectangle. (At the ends of the major axis the possibility of the membrane buckling is restrained by its smaller width, measured perpendicular to the major axis.)

(3) If the cross section has the form of a very narrow rectangle where side b is considerably smaller than side c (Fig. 88), then at points remote from the short sides the surface of the corresponding membrane can be assumed to be approximately cylindrical; in view of the uniformity of the load p on the membrane, the section of this surface by a plane perpendicular to the y axis will be a parabola [1]

$$z = \frac{p}{2H}\left(-\frac{b^2}{4} + x^2\right).$$

But, according to Prandtl's analogy [see formulas (8.73) and (8.74)], we have

$$\Phi = \mu\tau\left(\frac{b^2}{4} - x^2\right). \tag{8.77}$$

[1] Which is a funicular curve for a uniform load.

Further, according to (8.20), we get

$$M_t = -2c\mu\tau \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{b^2}{4} - x^2\right) dx = -\mu\tau \frac{b^3c}{3};$$

$$\tau = \frac{M_t}{\mu b^3 c \frac{1}{3}},$$

which has already been obtained previously. Consider in the cross section a straight line $y=\text{const}$, parallel to the x axis, which is drawn far from the ends of the section. The shearing stresses

$$Y_z = -\frac{\partial \Phi}{\partial x} = \frac{6M_t}{b^3 c} x$$

are distributed according to a linear law along this straight line (Fig. 88); the shearing stresses X_z will apparently be negligible far from the short sides of the rectangle.

Obviously, these results obtained for a narrow rectangle may be applied to other sections having the form of a bent strip (channels, I-beams, angles, unclosed rings, etc.) excepting, of course, those points where the curvature of the contour of the section changes abruptly; the configuration of the horizontals of the membrane becomes more complicated here (for instance, near the fillets and re-entrant corners of the section). These applications of Saint-Venant's theory and Prandtl's analogy are treated in detail in the courses on strength of materials and we shall not pursue the matter here.

In the case of a closed ring section one hits upon a feature characteristic of any doubly connected or, in general, multiply connected section bounded by several closed contours which do not intersect each other. The differential equation (8.67) and the boundary condition (8.68) for the stress function Φ hold good, but the constants C in (8.68) on each of the contours bounding the cection will be different; for one of the contours, e.g., for the outer contour, this constant can be chosen arbitrarily; the method of determining the remaining constants was indicated by R. Bredt [1]. Bredt's corresponding theorem, which follows from the general Stokes's formula, can readily be obtained for our purposes from Prandtl's analogy. In the case of a multiply connected section this

[1] See A. Föppl and L. Föppl, *Drang und Zwang*, 2. Aufl., Bd. 2, Berlin. 1928.

analogy has to be constructed as follows (Fig. 89). The regions F_1 and F_2 of the membrane, corresponding to the cavities in the section of a twisted bar, are covered by absolutely rigid plates glued to the membrane; after that a uniform pressure p is applied on the whole of the section F.

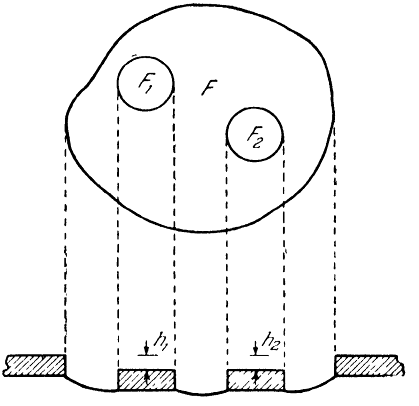

Fig. 89

Condition (8.68) requires that plates F_1 and F_2 move downward parallel to their initial position (for that, generally speaking, it is necessary to apply, in addition to the pressure, certain

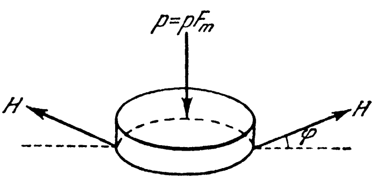

Fig. 90

couples to the plates; these, however, will not be involved in subsequent considerations). Let us write down the condition of equilibrium of one such plate F_m equating to zero the sum of the projections of the forces applied to it on the axis normal to the initial plane of the membrane. This condition will be of the form given by the following equation (Fig. 90):

$$\int H \sin \varphi \, ds = pF_m, \tag{8.78}$$

where the integral is extended over the entire contour of the plate concerned, and F_m designates its area. We observe that [cf. (8.76)]

$$\sin\varphi \approx \tan\varphi = \frac{\partial\Phi}{\partial v} = \operatorname{grad}\Phi = t.$$

Furthermore, we put H, as a constant, before the integral sign and bear in mind that according to (8.73) $p = -2\mu\tau H$. The equilibrium equation (8.78) becomes

$$\left.\begin{aligned} &\int \frac{\partial\Phi}{\partial v}\,ds = -2\mu\tau F_m \\ \text{or}\qquad &\int t\,ds = -2\mu\tau F_m; \end{aligned}\right\} \tag{8.79}$$

this is the required particular case of Bredt's theorem of "stress circulation" expressed by the integral in the left-hand member of the second of equations (8.79). We can write as many equations of this kind as there are inner cavities in a given section. It may be shown that the left-hand members of equations (8.79) depend linearly on the values of function Φ on the inner contours of the section; these values can be determined from (8.79). It is to be noted that in the case of a multiply connected contour formula (8.18), upon integration of the right-hand member by parts, yields the following result in place of (8.20):

$$M_t = 2C_0F_0 - 2\sum C_mF_m - 2\int\int \Phi\,ds, \tag{8.80}$$

where F_0 is the area bounded by the outer contour; C_0 is the value of function F_0 on this contour; F_m and C_m denote the same quantities for the inner contours. The double integral is extended over the area of the section minus the inner cavities. There remains in (8.80) to substitute the values of the constants C_0 and C_m obtained as indicated above.

55. CASE OF TRANSVERSE BENDING

Consider (Fig. 91) the case of the bending of a straight rod with an arbitrary cross section, of length l, built-in at the left-hand end and loaded at the right-hand end by a force Q which we assume to be applied in the form of shearing stresses distributed over the end cross section.

This problem was solved by Saint-Venant by the use of the semi-inverse method; we shall carry out its solution in terms of stresses by applying the results of Sec. 36. In solving a similar

problem in the case of plane strain or plane stress (Sec. 41), we have made the following assumptions:

$$Z_z = \frac{M}{J} x = -\frac{Q(l-z)}{J} x; \quad Y_z = Y_y = X_x = X_y = 0.$$

Following Saint-Venant, we shall also use the semi-inverse method to solve the present problem, though modifying slightly the assumptions by considering that the stress Y_z is different from zero as well. Consequently, we make the following assumptions:

$$\left.\begin{aligned} Z_z &= -\frac{Q(l-z)}{J} x, \\ X_x &= Y_y = X_y = 0. \end{aligned}\right\} \tag{8.81}$$

The remaining stresses $X_z = Z_x$ and $Y_z = Z_y$ will be found from the surface conditions so as to satisfy equations (I) and (VII) as

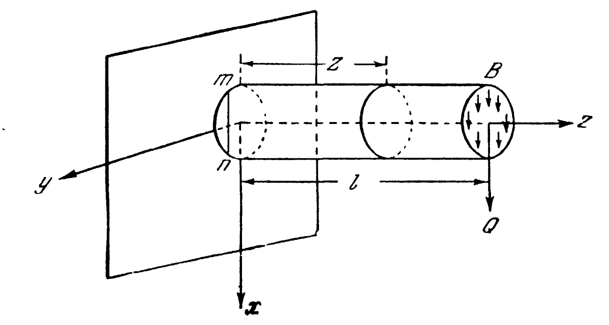

Fig. 91

well. Let us see what form these equations assume under our assumptions (8.81). The equilibrium equations (I) will be written as

$$\frac{\partial X_z}{\partial z} = 0; \quad \frac{\partial Y_z}{\partial z} = 0; \tag{8.82}$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} = -\frac{Qx}{J}. \tag{8.83}$$

Equations (8.82) show that the shearing stresses X_z and Y_z are independent of z, i.e., they must be distributed identically on all cross sections and, of course, they must be distributed on the end cross section B according to the same law. This restricts the manner in which the force Q may be applied at the section B. The force Q, when applied in a different manner, will produce other stresses; according to Saint-Venant's principle, however, the

difference in stress distribution will be essential only near the loaded end B.

We now proceed to equations (VII), observing first that in our case

$$\Theta = Z_z = -\frac{Q(l-z)}{J}x.$$

It can easily be seen that the first, second, third and sixth of equations (VII) are satisfied identically by virtue of assumptions (8.81). The fourth and fifth equations will be written as

$$(1+\sigma)\nabla^2 Y_z = 0; \quad (1+\sigma)\nabla^2 X_z + \frac{Q}{J} = 0$$

or

$$\nabla^2 Y_z = 0; \quad \nabla^2 X_z = -\frac{Q}{(1+\sigma)J}. \tag{8.84}$$

Let us also write down conditions (II) on the lateral surface of the rod. The first two are satisfied identically since $\cos(\nu z) = 0$ on the lateral surface. There remains only the last equation

$$Z_x \cos(\nu x) + Z_y \cos(\nu y) = 0. \tag{8.85}$$

Thus, the problem is reduced to the determination of two stresses X_z and Y_z, which are functions of x and y, from the condition that they satisfy equations (8.83) and (8.84) everywhere in the rod and (8.85) on its lateral surface.

Subsequently it will also be necessary to require that these shearing stresses on the end section reduce to the given load Q applied at the centroid of the section. As previously, we can simplify the solution of the problem by introducing the stress function; however, it is found convenient here to introduce two new functions; let the stresses be prescribed as

$$\left.\begin{aligned} Z_x = X_z &= \frac{\partial\varphi}{\partial y} - \frac{Qx^2}{2J} + f(y), \\ Z_y = Y_z &= -\frac{\partial\varphi}{\partial x}. \end{aligned}\right\} \tag{8.86}$$

Here $\varphi(x, y)$ and $f(y)$ are as yet arbitrary functions [1]. Substituting these expressions in equation (8.83), we see that it will be satisfied identically. It remains to satisfy equations (8.84) and the condition on the lateral surface (8.85).

Substituting stresses (8.86) in equations (8.84), we write them down as

$$\frac{\partial}{\partial x}\nabla^2\varphi = 0; \quad \frac{\partial}{\partial y}\nabla^2\varphi - \frac{Q}{J} + f''(y) = -\frac{Q}{J(1+\sigma)}$$

[1] The introduction of the function $f(y)$ was proposed by S. P. Timoshenko.

or

$$\frac{\partial}{\partial x}\nabla^2\varphi=0;\quad \frac{\partial}{\partial y}\nabla^2\varphi=\frac{\sigma Q}{J(1+\sigma)}-f''(y).$$

From these equations we easily find

$$\nabla^2\varphi=\frac{Q\sigma y}{J(1+\sigma)}-f'(y)+C. \tag{8.87}$$

We proceed to the condition on the lateral surface (8.85). As in the torsion problem, we obtain

$$\cos(vx)=\frac{dy}{ds};\quad \cos(vy)=-\frac{dx}{ds}.$$

Substituting this and (8.86) in condition (8.85), we have

$$\left[\frac{\partial\varphi}{\partial y}-\frac{Qx^2}{2J}+f(y)\right]\frac{dy}{ds}+\frac{\partial\varphi}{\partial x}\frac{dx}{ds}=0$$

or, bearing in mind that

$$\frac{\partial\varphi}{\partial x}\frac{dx}{ds}+\frac{\partial\varphi}{\partial y}\frac{dy}{ds}=\frac{d\varphi}{ds},$$

we get

$$\frac{d\varphi}{ds}=\left[\frac{Qx^2}{2J}-f(y)\right]\frac{dy}{ds}. \tag{8.88}$$

The question is reduced to the integration of equation (8.87) subject to condition (8.88) on the lateral surface. Let us try to adjust the arbitrary function $f(y)$, introduced here, to make the bracketed expression in equation (8.88) varnish on the contour of the cross section, i.e., the following condition must be fulfilled on the contour:

$$f(y)-\frac{Qx^2}{2J}=0; \tag{8.89}$$

equation (8.88) will then be reduced to the condition $\frac{d\varphi}{ds}=0$ requiring that the stress function φ remain the same on the contour of the section. It remains to consider the condition on the end cross section. By projecting the forces applied to this section on the axis Oy we have

$$\int\int Y_z\,dx\,dy=-\int\int\frac{\partial\varphi}{\partial x}dx\,dy=-\int dy\int\frac{\partial\varphi}{\partial x}dx=0,$$

i.e., the sum of the tangential tractions Q acts in a vertical direction (Fig. 91). Furthermore,

$$\int\int X_z\,dx\,dy = \int\int \frac{\partial\varphi}{\partial y}\,dx\,dy - \frac{Q}{2J}\int\int x^2\,dx\,dy + \int\int f(y)\,dx\,dy =$$

$$= -\frac{Q}{2} + \int dy \int f(y)\,dx = -\frac{Q}{2} + \int dy\,[f(y_p)\,x_p - f(y_m)\,x_m] =$$

$$= -\frac{Q}{2} + \int dy\left[\frac{Qx_p^3}{2J} - \frac{Qx_m^3}{2J}\right] =$$

$$= -\frac{Q}{2} + \frac{3}{2}\frac{Q}{J}\int\frac{\left(x_p^3 - x_m^3\right)dy}{3} = -\frac{Q}{2} + \frac{3}{2}\frac{Q}{J} = Q.$$

Thus, the shearing stresses on the end section do in fact reduce to the given load Q directed downward.

We have still one arbitrary constant C left in equation (8.87). It is easy to check that for a section which is symmetrical with respect to the plane xOz one should take $C=0$.

To solve the problem it is finally necessary to choose $f(y)$ so as to satisfy equation (8.89) on the contour. We introduce this expression of $f(y)$ in equation (8.87) and integrate it. Having thus obtained $\varphi(x, y)$ we introduce it in equation (8.86) and find the shearing stresses.

If the cross section of a rod is not symmetrical with respect to the principal centroidal axis Ox, the phenomenon of bending becomes far more involved. The solution of the problem obtained here holds true if the resultant of the load Q lies not in the principal plane Oxz of the rod but in another plane parallel to it and intersecting the cross section at a point called flexural centre. In the case of thin-walled rods the flexural centre can be found approximately by the elementary method given in the courses on strength of materials.

Example. Bending of a circular rod. Let the contour of the section be a circle:

$$x^2 + y^2 - r^2 = 0. \tag{8.90}$$

To satisfy equation (8.89) on the contour one can obviously prescribe the function $f(y)$ as

$$f(y) = \frac{Q}{2J}(r^2 - y^2). \tag{8.91}$$

After that equation (8.87) becomes

$$\nabla^2\varphi = \frac{\sigma Q y}{J(1+\sigma)} + \frac{Qy}{J} + C = \frac{1+2\sigma}{1+\sigma}\frac{Q}{J}\,y + C. \tag{8.92}$$

Let us attempt to satisfy this equation by prescribing the function φ in the following form:

$$\varphi = m(r^2 - x^2 - y^2)\,y.$$

For substitution in equation (8.92) we have

$$\frac{\partial^2\varphi}{\partial x^2} = -2my; \qquad \frac{\partial^2\varphi}{\partial y^2} = -6my,$$

and equation (8.92) yields

$$-8my = \frac{1+2\sigma}{1+\sigma}\,\frac{Q}{J}\,y + C.$$

Hence, obviously, $C=0$;

$$m = -\frac{1}{8}\,\frac{(1+2\sigma)}{(1+\sigma)}\,\frac{Q}{J}.$$

The stress function is finally found to be

$$\varphi = \frac{1}{8}\,\frac{1+2\sigma}{1+\sigma}\,\frac{Q}{J}\,(x^2+y^2-r^2)\,y.$$

Substituting this expression of the function φ and also $f(y)$ from equation (8.91) in equations (8.86), we find the shearing stresses

$$\left.\begin{aligned} X_z &= \frac{Q}{2J}\,\frac{3+2\sigma}{4(1+\sigma)}\left(r^2 - x^2 - \frac{1-2\sigma}{3+2\sigma}\,y^2\right), \\ Y_z &= -\frac{1}{4}\,\frac{1+2\sigma}{1+\sigma}\,\frac{Q}{J}\,xy. \end{aligned}\right\} \tag{8.93}$$

We observe that the shearing stresses depend on Poisson's ratio σ; assuming, for instance, that $\sigma = \frac{1}{3}$, we get

$$\left.\begin{aligned} X_z &= \frac{11}{32}\,\frac{Q}{J}\left(r^2 - x^2 - \frac{1}{11}\,y^2\right), \\ Y_z &= -\frac{5}{16}\,\frac{Q}{J}\,xy. \end{aligned}\right\} \tag{8.94}$$

Since, in general, $Y_z \neq 0$, the total shearing stress

$$t = \sqrt{X_z^2 + Y_z^2}$$

is acting in a direction other than vertical. At the points of the section which are located on the co-ordinate axes $(x=0,\ y=0)$ the component Y_z is absent and t is vertical here. On the neutral axis $(x=0)$ we have

$$X_z = \frac{Q}{2J}\,\frac{3+2\sigma}{4(1+\sigma)}\left(r^2 - \frac{1-2\sigma}{3+2\sigma}\,y^2\right).$$

It follows from this that the stresses X_z are distributed nonuniformly along the neutral axis. Their maximum value (at the centre of the section for $y=0$) is

$$X_z = \frac{Q}{2J}\,\frac{3+2\sigma}{4(1+\sigma)}\,r^2;$$

for $\sigma = \frac{1}{3}$ we find from equations (8.94)

$$\max X_z = \frac{11}{32}\,\frac{Q}{J}\,r^2 = \frac{11}{32}\,\frac{Q4r^2}{\pi r^4} = \frac{11}{8}\,\frac{Q}{F} = 1.375\,\frac{Q}{F}, \qquad (8.95)$$

where $F=\pi r^2$ is the cross-sectional area of the rod. The elementary calculation according to the formula $X_z = t = \frac{QS}{Jb}$ gives

$$\max X_z = \frac{4Q}{3\pi r^2} = 1.33\,\frac{Q}{F}.$$

The error, as compared to the exact value (8.95), is about 3.4 per cent. The stresses Y_z are usually neglected in the elementary analysis.

IX

More General Methods of Solving Elasticity Problems

56. GENERAL SOLUTION OF DIFFERENTIAL EQUATIONS OF EQUILIBRIUM IN TERMS OF STRESSES. STRESS FUNCTIONS

The solution of problems of the theory of elasticity in terms of stresses requires the simultaneous solution of two systems of differential equations: the equations of equilibrium (I) and the Beltrami-Michell compatibility equations (VII). We shall limit ourselves to the case where body forces are absent; then all these equations will be homogeneous. In this section it will be shown that the system of equilibrium equations

$$\left.\begin{aligned}\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0,\\ \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}=0,\\ \frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}=0\end{aligned}\right\} \tag{9.1}$$

can be solved in the general form, all the components of the stress tensor being expressed in terms of three arbitrary functions of x, y, z, called *stress functions*; we have already encountered similar solutions (though for narrower classes of problems) in the plane problem (Airy's function) and in Saint-Venant's theory of torsion (Prandtl's function).

If the general solution of the equations of equilibrium (9.1) has been obtained, it will no longer be necessary to deal with them in future and it will remain to integrate only Beltrami's equations; this will facilitate to some extent the solution of the problem.

One of the forms of the general solution was proposed by J. C. Maxwell in 1862; another form was given by G. Morera in 1892. These solutions can be obtained without great difficulty.

1. *Maxwell's solution.* Let us take three arbitrary functions

$$\varphi_1(x,\ y,\ z);\quad \varphi_2(x,\ y,\ z);\quad \varphi_3(x,\ y,\ z), \tag{9.2}$$

and express the shearing stresses through them as follows:

$$X_y = -\frac{\partial^2 \varphi_3}{\partial x\,\partial y};\quad Y_z = -\frac{\partial^2 \varphi_1}{\partial y\,\partial z};\quad Z_x = -\frac{\partial^2 \varphi^2}{\partial z\,\partial x}. \tag{9.3}$$

These formulas are obtained from one another by cyclic changes between

$$(x,\ y,\ z);\quad (\varphi_1,\ \varphi_2,\ \varphi_3).$$

We shall see later that there are other possible ways of prescribing the shearing stresses, but in any case their formulas should not be altered in form in interchanging the letters in the notation of stresses, as required by the law of reciprocity of shearing stresses, for instance

$$X_y = -\frac{\partial^2 \varphi_3}{\partial x\,\partial y} = -\frac{\partial^2 \varphi_3}{\partial y\,\partial x} = Y_x.$$

Substituting the expressions of X_y and X_z from (9.3) in the first of equations (9.1), we obtain

$$\frac{\partial X_x}{\partial x} = \frac{\partial^3 \varphi_3}{\partial x\,\partial y^2} + \frac{\partial^2 \varphi_2}{\partial z^2\,\partial x}.$$

Hence we find the component X_x by integrating with respect to x and omitting an arbitrary constant of integration (i.e., an arbitrary function of y and z) which is of no importance since the functions φ_2 and φ_3 are arbitrary:

$$X_x = \frac{\partial^2 \varphi_3}{\partial y^2} + \frac{\partial^2 \varphi_2}{\partial z^2}. \tag{9.4'}$$

In the same way we obtain the remaining normal components Y_y and Z_z from the second and third equations of (9.1); their expressions, however, can be obtained from (9.4′) by the above-indicated cyclic changes. As a result, we have the following solution of equations (9.1):

$$\left.\begin{aligned}
X_x &= \frac{\partial^2 \varphi_3}{\partial y^2} + \frac{\partial^2 \varphi_2}{\partial z^2}; & X_y &= -\frac{\partial^2 \varphi_3}{\partial x\,\partial y};\\
Y_y &= \frac{\partial^2 \varphi_1}{\partial z^2} + \frac{\partial^2 \varphi_3}{\partial x^2}; & Y_z &= -\frac{\partial^2 \varphi_1}{\partial y\,\partial z};\\
Z_z &= \frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2}; & Z_x &= -\frac{\partial^2 \varphi_2}{\partial z\,\partial x}.
\end{aligned}\right\} \tag{9.4}$$

Functions (9.2) in this solution are called *Maxwell's stress functions.*

That solution (9.4) in general will be confirmed, if we prove that for any tensor

$$\begin{pmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{pmatrix}, \tag{9.5}$$

which satisfies the equations of equilibrium (9.1), it is possible to construct three functions (9.2). This can easily be done by integrating twice the equalities in the right-hand column of (9.4):

$$\varphi_3 = -\int\int X_y\,dx\,dy; \quad \varphi_1 = -\int\int Y_z\,dy\,dz;$$

$$\varphi_2 = -\int\int X_z\,dx\,dz.$$

The constants of integration are omitted, as above. It remains to ascertain that when these values of functions are substituted in the equalities of the left-hand column of (9.4) the latter become identities; for substitution in the first equality we find

$$\frac{\partial^2\varphi_3}{\partial y^2} = -\int \frac{\partial X_y}{\partial y}\,dx; \quad \frac{\partial^2\varphi_2}{\partial z^2} = -\int \frac{\partial X_z}{\partial z}\,dx;$$

then it will be written down as

$$X_x = -\int \left(\frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z}\right) dx.$$

After differentiating it with respect to x, we find

$$\frac{\partial X_x}{\partial x} = -\frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z}.$$

On the basis of the first of equations (9.1) this equality gives an identity; in the same way we can prove that the other two equalities (9.4) become identities.

2. *Morera's solution.* Let us take three arbitrary functions

$$\psi_1(x, y, z); \quad \psi_2(x, y, z); \quad \psi_3(x, y, z), \tag{9.6}$$

and express the normal stresses through them:

$$X_x = \frac{\partial^2\psi_1}{\partial y\,\partial z}; \quad Y_y = \frac{\partial^2\psi_2}{\partial z\,\partial x}; \quad Z_z = \frac{\partial^2\psi_3}{\partial x\,\partial y}. \tag{9.7}$$

These formulas are also obtained one from another by cyclic changes between

$$(x,\ y,\ z);\quad (\psi_1,\ \psi_2,\ \psi_3).$$

Introducing the values of stresses (9.7) in the equations of equilibrium (9.1), we write them down as

$$\left.\begin{array}{r|l}
+\dfrac{\partial}{\partial x} & \dfrac{\partial X_y}{\partial y}+\dfrac{\partial X_z}{\partial z}=-\dfrac{\partial^3\psi_1}{\partial x\,\partial y\,\partial z}, \\
+\dfrac{\partial}{\partial y} & \dfrac{\partial X_y}{\partial x}+\dfrac{\partial Y_z}{\partial z}=-\dfrac{\partial^3\psi_2}{\partial x\,\partial y\,\partial z}, \\
-\dfrac{\partial}{\partial z} & \dfrac{\partial X_z}{\partial x}+\dfrac{\partial Y_z}{\partial y}=-\dfrac{\partial^3\psi_3}{\partial x\,\partial y\,\partial z}.
\end{array}\right\} \tag{9.8}$$

After differentiating them according to the scheme indicated on the left and adding them up by members, we obtain

$$2\frac{\partial^2 X_y}{\partial x\,\partial y}=-\frac{\partial^3}{\partial x\,\partial y\,\partial z}\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}-\frac{\partial\psi_3}{\partial z}\right);$$

whence we have

$$X_y=-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}-\frac{\partial\psi_3}{\partial z}\right).$$

The other two shearing components will be obtained by the above-mentioned cyclic changes or by a change of signs in differentiating equations (9.8). We thus arrive at the following form of the solution of equations (9.1):

$$\left.\begin{aligned}
X_x&=\frac{\partial^2\psi_1}{\partial y\,\partial z}; & X_y&=-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}-\frac{\partial\psi_3}{\partial z}\right);\\
Y_y&=\frac{\partial^2\psi_2}{\partial z\,\partial x}; & Y_z&=-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi_2}{\partial y}+\frac{\partial\psi_3}{\partial z}-\frac{\partial\psi_1}{\partial x}\right);\\
Z_z&=\frac{\partial^2\psi_3}{\partial x\,\partial y}; & Z_x&=-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial\psi_3}{\partial z}+\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_2}{\partial y}\right).
\end{aligned}\right\} \tag{9.9}$$

Functions (9.6) in this solution are called *Morera's stress functions*. The generality of Morera's solution can be proved by the method described above for Maxwell's solution; functions (9.6) are found by the double integration of the equalities in the left-hand column of (9.9); their substitution in the equalities of the right-hand column shows that they become identities on the basis of equilibrium equations (9.1).

Maxwell's and Morera's solutions are not the only possible ones; V. I. Blokh and Ju. A. Krutkov indicated a number of other forms of the general solution of the equilibrium equations [1]. These solutions are expressed in terms of the second derivatives of stress functions; it is possible to seek solutions expressed in terms of derivatives of higher orders. Let us take, for instance, one arbitrary function $F(x, y, z)$ and three constant parameters a, b, c, also arbitrary, and write down the components of the stress tensor in the following form:

$$\left.\begin{aligned} X_x &= (b+c)\frac{\partial^4 F}{\partial y^2\,\partial z^2}; & X_y &= -c\,\frac{\partial^4 F}{\partial x\,\partial y\,\partial z^2};\\ Y_y &= (c+a)\frac{\partial^4 F}{\partial z^2\,\partial x^2}; & Y_z &= -a\,\frac{\partial^4 F}{\partial y\,\partial z\,\partial x^2};\\ Z_z &= (a+b)\frac{\partial^4 F}{\partial x^2\,\partial y^2}; & Z_x &= -b\,\frac{\partial^4 F}{\partial z\,\partial x\,\partial y^2}. \end{aligned}\right\} \tag{9.10}$$

Substituting these values of the components in the equations of equilibrium (9.1), we see that they are identically satisfied. Let us set up a sum of solutions of the form (9.10):

$$\left.\begin{aligned} X_x &= \sum_{m=1}^{\infty}(b_m+c_m)\frac{\partial^4 F_m}{\partial y^2\,\partial z^2}; & X_y &= -\sum_{m=1}^{\infty} c_m\,\frac{\partial^4 F_m}{\partial x\,\partial y\,\partial z^2};\\ Y_y &= \sum_{m=1}^{\infty}(c_m+a_m)\frac{\partial^4 F_m}{\partial z^2\,\partial x^2}; & Y_z &= -\sum_{m=1}^{\infty} a_m\,\frac{\partial^4 F_m}{\partial y\,\partial z\,\partial x^2};\\ Z_z &= \sum_{m=1}^{\infty}(a_m+b_m)\frac{\partial^4 F_m}{\partial x^2\,\partial y^2}; & Z_x &= -\sum_{m=1}^{\infty} b_m\,\frac{\partial^4 F_m}{\partial z\,\partial x\,\partial y^2}; \end{aligned}\right\} \tag{9.11}$$

it will also be a solution of equations (9.1).

If we now put

$$\sum_{m=1}^{\infty} a_m\frac{\partial^2 F_m}{\partial x^2} = \varphi_1(x, y, z); \quad \sum_{m=1}^{\infty} b_m\frac{\partial^2 F_m}{\partial y^2} = \varphi_2(x, y, z);$$
$$\sum_{m=1}^{\infty} c_m\frac{\partial^2 F_m}{\partial z^2} = \varphi_3(x, y, z) \tag{9.12}$$

and substitute these values in formulas (9.11), we shall obtain Maxwell's solution (9.4). Since the parameters a_m, b_m, c_m are

[1] Ju. A. Krutkov, *The Stress-function Tensor and General Solutions in Statics of the Theory of Elasticity.* Moscow, Izd. AN SSSR, 1949; V. I. Blokh, Prikl. matem. i mekhan., 1950, Vol 14, No 4, pp. 415-422

arbitrary, functions (9.12) are independent of each other, and since the functions $F_m(x, y, z)$ are arbitrary, they are also arbitrary.

Let us now replace the parameters a_m, b_m, c_m by others according to the formulas

$$\left.\begin{aligned} b_m+c_m&=\alpha_m; & a_m&=\frac{1}{2}(\beta_m+\gamma_m+\alpha_m);\\ c_m+a_m&=\beta_m; & b_m&=\frac{1}{2}(\gamma_m+\alpha_m+\beta_m);\\ a_m+b_m&=\gamma_m; & c_m&=\frac{1}{2}(\alpha_m+\beta_m+\gamma_m) \end{aligned}\right\} \tag{9.13}$$

and set

$$\sum_{m=1}^{\infty}\alpha_m\frac{\partial^2 F_m}{\partial y\,\partial z}=\psi_1(x, y, z);\quad \sum_{m=1}^{\infty}\beta_m\frac{\partial^2 F_m}{\partial z\,\partial x}=\psi_2(x, y, z);$$

$$\sum_{m=1}^{\infty}\gamma_m\frac{\partial^2 F_m}{\partial x\,\partial y}=\psi_3(x, y, z). \tag{9.14}$$

Substituting these values in formulas (9.11), we obtain Morera's solution (9.9). Formulas (9.12) and (9.14) present Maxwell's and Morera's functions in the form of series; consequently, the parameters a_m, b_m, c_m and α_m, β_m, γ_m entering in them should be chosen so that the series converge and the formulas make sense.

It will be observed that if we set $\varphi_1=\varphi_2\equiv 0$ in Maxwell's formulas (9.4) and if the function $\varphi_3=\varphi(x, y)$ (i.e., it does not depend on z), it reprecents Airy's function in the plane problem and formulas (9.4) coincide with formulas (6.16) of Chapter VI.

57. EQUATIONS OF EQUILIBRIUM IN CYLINDRICAL CO-ORDINATES. THEIR GENERAL SOLUTION

The cylindrical co-ordinates have been employed in Sec. 47 for the solution of the plane problem; by using them, we have obtained the differential equations of equilibrium (Ipp) which do not include, however, certain components of the stress tensor corresponding to the general three-dimensional problem of the theory of elasticity. Fig. 92 shows these components:

$$Z_z;\quad R_z=Z_r;\quad Z_\theta=\Theta_z. \tag{9.15}$$

To derive the differential equations of the three-dimensional problem let us isolate from a body the element shown in Fig. 65 (it is repeated in Fig. 92). Let us apply to it the stresses involved in the plane problem (Fig. 66) and those which we have introduced anew; it is necessary, of course, to take into account the increments of components (9.15), which they receive on account of the increments of co-ordinates dr, $d\theta$, dz in passing from one face of the element to another, as shown in Fig. 66 in respect to the components of the plane problem. By projecting all forces applied to the element on the three axes we obtain three differential equations of equilibrium which represent a generalisation of equations (Ipp) of Sec. 47 of the plane problem:

$$\left.\begin{aligned}
&\frac{\partial R_r}{\partial r}+\frac{1}{r}\frac{\partial R_\theta}{\partial \theta}+\frac{\partial R_z}{\partial z}+\frac{R_r-\Theta_\theta}{r}=0,\\
&\frac{\partial \Theta_r}{\partial r}+\frac{1}{r}\frac{\partial \Theta_\theta}{\partial \theta}+\frac{\partial \Theta_z}{\partial z}+\frac{2R_\theta}{r}=0,\\
&\frac{\partial Z_r}{\partial r}+\frac{1}{r}\frac{\partial Z_\theta}{\partial \theta}+\frac{\partial Z_z}{\partial z}+\frac{R_z}{r}=0.
\end{aligned}\right\} \tag{9.16}$$

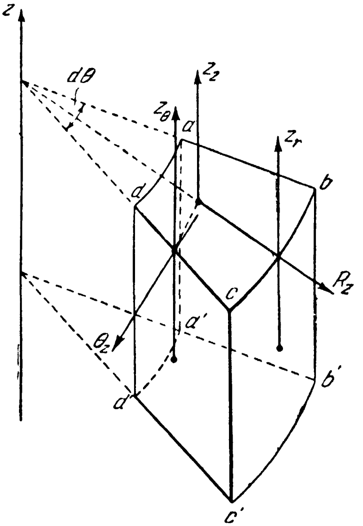

Fig. 92

The first three terms of each of these equations can be formally obtained from equations (9.1) by replacing the notations:

x by r; $\quad \partial x$ by ∂r;

y by θ; $\quad \partial y$ by $r\,\partial\theta$.

The last terms, containing the components themselves, are present because of the curvilinearity of co-ordinate surfaces $r=\text{const}$ (cf. Fig. 64); because of this the area $add'a'$ (Fig. 92) is not equal to the area $bcc'b'$, and the areas $cdd'c'$ and $abb'a'$ are not parallel to each other; these circumstances and their effect can be traced in the derivation of equations (Ipp) in Sec. 47.

As for equations (9.1), one can obtain the general solution for equations (9.16), its forms being still more various than in the case of Cartesian co-ordinates (see the afore-mentioned works of V. I. Blokh and Ju. A. Krutkov). The general solution also contains three arbitrary stress functions. We present one of the possible forms of the solution [1] expressed in terms of the stress func-

[1] See the author's paper, *Some Generalisations of Lamé's Problem for an Elastic Parallelepiped*, Prikl. matem. i mekhan., 1953, Vol. 17, No. 4, pp. 465-469.

tions:

$$f_1(r,\ \theta,\ z);\quad f_2(r,\ \theta,\ z);\quad f_3(r,\ \theta,\ z); \tag{9.17}$$

$$\left.\begin{aligned}
&R_r=\frac{1}{r}\frac{\partial^2 f_1}{\partial z^2};\quad \Theta_\theta=\frac{\partial^2 f_2}{\partial z^2};\quad R_\theta=\frac{1}{r}\frac{\partial^2 f_3}{\partial z^2};\\
&Z_z=\frac{1}{r^2}\frac{\partial^2 f_2}{\partial \theta^2}-\frac{1}{r}\frac{\partial f_2}{\partial r}+\frac{1}{r}\frac{\partial^2 f_1}{\partial r^2}+\frac{2}{r^2}\frac{\partial^2 f_3}{\partial r\,\partial\theta};\\
&R_z=\frac{1}{r}\frac{\partial f_2}{\partial z}-\frac{1}{r}\frac{\partial^2 f_1}{\partial r\,\partial z}-\frac{1}{r^2}\frac{\partial^2 f_3}{\partial\theta\,\partial z};\\
&Z_\theta=-\frac{1}{r}\frac{\partial^2 f_2}{\partial\theta\,\partial z}-\frac{1}{r^2}\frac{\partial f_3}{\partial z}-\frac{1}{r}\frac{\partial^2 f_3}{\partial r\,\partial z}.
\end{aligned}\right\} \tag{9.18}$$

The generality of this solution is proved, just as in Sec. 56, by determining the functions f_1, f_2, f_3 from the equations of the first line of (9.18) and by substituting them in the remaining equations. As in the case of the plane problem in polar co-ordinates, formulas (9.18) are considerably simplified if the state of stress is independent of the co-ordinate θ; then all the derivatives with respect to this co-ordinate vanish; we obtain

$$\left.\begin{aligned}
&R_r=\frac{1}{r}\frac{\partial^2 f_1}{\partial z^2};\quad \Theta_\theta=\frac{\partial^2 f_2}{\partial z^2};\quad R_\theta=\frac{1}{r}\frac{\partial^2 f_3}{\partial z^2};\\
&Z_z=-\frac{1}{r}\frac{\partial}{\partial r}\left(f_2-\frac{\partial f_1}{\partial r}\right);\quad R_z=\frac{1}{r}\frac{\partial}{\partial z}\left(f_2-\frac{\partial f_1}{\partial z}\right);\\
&Z_\theta=-\frac{1}{r^2}\frac{\partial^2 (rf_3)}{\partial r\,\partial z}.
\end{aligned}\right\} \tag{9.19}$$

However, a thorough examination of these formulas shows a possibility of further simplification. In fact, let us introduce three new stress functions in place of f_1, f_2, f_3:

$$\omega=\frac{\partial^2 f_1}{\partial z^2};\quad \psi=f_2-\frac{\partial f_1}{\partial r};\quad \chi=\frac{\partial (rf_3)}{\partial z},$$

whence we find

$$f_2=\psi+\frac{\partial f_1}{\partial r};\quad \frac{\partial^2 f_2}{\partial z^2}=\frac{\partial^2\psi}{\partial z^2}+\frac{\partial\omega}{\partial r},$$

and formulas (9.19) assume a very simple form:

$$\left.\begin{aligned}
&R_r=\frac{\omega}{r};\quad \Theta_\theta=\frac{\partial\omega}{\partial r}+\frac{\partial^2\psi}{\partial z^2};\quad R_\theta=\frac{1}{r^2}\frac{\partial\chi}{\partial z};\\
&R_z=\frac{1}{r}\frac{\partial\psi}{\partial z};\quad Z_z=-\frac{1}{r}\frac{\partial\psi}{\partial r};\quad Z_\theta=-\frac{1}{r^2}\frac{\partial\chi}{\partial r}.
\end{aligned}\right\} \tag{9.20}$$

These formulas enable us to distinguish two categories of problems which are important in the sense of practical applications.

1. Assume $\chi=0$; then $R_\theta=Z_\theta=0$; thus (Figs 66 and 92) there remain only $R_z=Z_r$ among the shearing stresses and we have a

state of *axisymmetrical three-dimensional stress*; but according to Hooke's law

$$e_{r\theta}=\frac{1}{\mu}R_{\theta}=0; \quad e_{z\theta}=\frac{1}{\mu}Z_{\theta}=0,$$

and, therefore, we have an axisymmetrical three-dimensional strain; the nonvanishing components of the stress tensor R_r, Θ_θ, Z_z and $R_z=Z_r$ are expressed in terms of two stress functions ω and ψ.

2. Assume $\omega=\psi=0$. In this case only two components of the stress tensor are different from zero:

$$R_{\theta}=\frac{1}{r^2}\frac{\partial\chi}{\partial z}; \quad Z_{\theta}=-\frac{1}{r^2}\frac{\partial\chi}{\partial r};$$

they are expressed in terms of a single stress function. This solution is employed in the problem of torsion of a rod having the form of a body of revolution.

The general solutions of the equations of equilibrium, presented in the last two sections, do not by themselves give the solution of the elasticity problem, since the stress functions involved in them must be determined from the compatibility conditions (for instance, from Beltrami's equations in Cartesian co-ordinates) and from the conditions on the surface of a body; these solutions, however, are of essential help in solving problems by the variational method elaborated by A. Castigliano; this method is discussed in Chapter XI with an appropriate use of the solutions in question.

58. HARMONIC AND BIHARMONIC FUNCTIONS

In the preceding chapters we encountered harmonic and biharmonic functions. It will be seen from what follows that the solution of all problems of the theory of elasticity is very closely bound up with these functions; in this section and the next we shall, therefore, acquire greater acquaintance of some properties of harmonic and biharmonic functions and the general methods of constructing the latter.

By a harmonic function

$$\varphi(x, y, z) \tag{9.21}$$

is meant, as is known, a function which satisfies Laplace's equation

$$\nabla^2\varphi=0. \tag{9.22}$$

Harmonic functions are sometimes called the Laplacian potentials or Laplace's potentials. Let us take a system of Cartesian co-ordinates xyz in space; then the field of a function $\varphi(x, y, z)$ is defined as that part of space at all points of which the function $\varphi(x, y, z)$ takes on particular finite values and remains single-valued. If the function φ is a potential, an additional requirement is made to it: it is necessary that its first partial derivatives exist within the field; one has often to require that derivatives of higher order exist as well; for instance, it can be seen from equation (9.22) that the derivatives of at least the second order must exist for Laplace's potential.

If all we require from function (9.21) is that it satisfy Laplace's equation, we notice that there exists an infinite number of such functions. For instance, if a certain harmonic function is known, then proceeding from it, we can obtain any number of new harmonic functions by making use of the property of Laplace's operator indicated in formula (5.54) of Sec. 36. Indeed, if $\varphi(x, y, z, a, b, c, \ldots)$ is a harmonic function and $a, b, c, \ldots$ are any parameters of coefficients entering in its expression, then one can easily show that the partial derivative

$$\frac{\partial\varphi}{\partial t}$$

where t is any of the quantities $x, y, z, a, b, c, \ldots$ is also a harmonic function. Let us verify this:

$$\nabla^2\left(\frac{\partial\varphi}{\partial t}\right)=\frac{\partial}{\partial t}\nabla^2\varphi;$$

but, since φ is a harmonic function, $\nabla^2\varphi=0$ and, consequently,

$$\nabla^2\left(\frac{\partial\varphi}{\partial t}\right)=0,$$

which was to be proved. It is clear that if there exist consecutive derivatives of the function

$$\frac{\partial^2\varphi}{\partial t^2},\quad \frac{\partial^3\varphi}{\partial t^3},\quad \text{etc.},$$

they are all harmonic functions, too.

By way of example, let us consider the function

$$\varphi=\frac{1}{\sqrt{x^2+y^2+z^2}},$$

or, more compactly,

$$\varphi=\frac{1}{r},\tag{9.23}$$

where $r=\sqrt{x^2+y^2+z^2}$ is the radius-vector from the origin to a point (x, y, z).

Let us first show that function (9.23) is harmonic. Note first that from the relation

$$r^2=x^2+y^2+z^2 \tag{9.24}$$

we obtain

$$\frac{\partial (r^2)}{\partial x}=2r\frac{\partial r}{\partial x}=2x, \tag{9.25}$$

or

$$\frac{\partial r}{\partial x}=\frac{x}{r};$$

in the same way

$$\frac{\partial r}{\partial y}=\frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z}=\frac{z}{r}. \tag{9.26}$$

On this basis we obtain from (9.23)

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}=-\frac{1}{r^2}\frac{x}{r}=-\frac{x}{r^3},$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)=\frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right)=-\frac{1\cdot r^3-3r^2\frac{\partial r}{\partial x}x}{r^6}=-\frac{1}{r^3}+\frac{3x^2}{r^5}. \tag{9.27}$$

Similarly, differentiating with respect to y and z, we have

$$\left.\begin{aligned}\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right)&=-\frac{1}{r^3}+3\frac{y^2}{r^5},\\ \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)&=-\frac{1}{r^3}+3\frac{z^2}{r^5}.\end{aligned}\right\} \tag{9.28}$$

Adding these equalities by members, we find

$$\nabla^2\left(\frac{1}{r}\right)=-3\frac{1}{r^3}+3\frac{x^2+y^2+z^2}{r^5}=-3\frac{1}{r^3}+3\frac{1}{r^3}=0,$$

which was to be proved. Hence function (9.23) is Laplace's potential. The field of this function is the entire unbounded space except for a single point—the origin, where function (9.23) becomes infinite.

The gradient of function (9.23) has the following projections on the co-ordinate axes [cf. formulas (9.27)]:

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=-\frac{x}{r^3}; \quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right)=-\frac{y}{r^3};\\ &\frac{\partial}{\partial z}\left(\frac{1}{r}\right)=-\frac{z}{r^3}.\end{aligned}\right\} \tag{9.29}$$

However, we observe that

$$\frac{x}{r}=\cos(r,\ x);\quad \frac{y}{r}=\cos(r,\ y);\quad \frac{z}{r}=\cos(r,\ z),$$

and then formulas (9.29) can be rewritten as

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)&=-\frac{1}{r^2}\cos(r,\ x),\\ \frac{\partial}{\partial y}\left(\frac{1}{r}\right)&=-\frac{1}{r^2}\cos(r,\ y),\\ \frac{\partial}{\partial z}\left(\frac{1}{r}\right)&=-\frac{1}{r^2}\cos(r,\ z).\end{aligned}\right\}\tag{9.30}$$

Hence we conclude that

(1) the magnitude of the gradient is $\frac{1}{r^2}$;

(2) it is directed along the radius-vector r towards the origin. This kind of field is obtained if we take the origin to be a centre which attracts a material particle according to the law of universal gravitation (inversely proportional to the square of the distance).

As the function $\frac{1}{r}$ is a harmonic function, by differentiating it we obtain a series of new harmonic functions. For instance, equations (9.29) give us three harmonic functions:

$$-\frac{x}{r^3},\quad -\frac{y}{r^3},\quad -\frac{z}{r^3}.\tag{9.31}$$

Functions (9.28) are obviously harmonic as well.

Let us prove now a lemma which will be very useful for what follows: if $\psi(x,\ y,\ z)$ is a harmonic function, we have the identity

$$\nabla^2(x\psi)=2\frac{\partial\psi}{\partial x};\tag{9.32}$$

indeed, by successive differentiation of the function $x\psi$ we obtain

$$\left.\begin{aligned}&\frac{\partial}{\partial x}(x\psi)=\psi+x\frac{\partial\psi}{\partial x}, && \frac{\partial^2}{\partial x^2}(x\psi)=2\frac{\partial\psi}{\partial x}+x\frac{\partial^2\psi}{\partial x^2},\\ &\frac{\partial}{\partial y}(x\psi)=x\frac{\partial\psi}{\partial y}, && \frac{\partial^2}{\partial y^2}(x\psi)=x\frac{\partial^2\psi}{\partial y^2},\\ &\frac{\partial}{\partial z}(x\psi)=x\frac{\partial\psi}{\partial z}, && \frac{\partial^2}{\partial z^2}(x\psi)=x\frac{\partial^2\psi}{\partial z^2}.\end{aligned}\right\}\tag{9.33}$$

Adding up equalities (9.33) by members and observing that $\nabla^2\psi=0$ according to condition, we see the validity of (9.32). This lemma can obviously be extended as follows: if ψ_1, ψ_2, ψ_3

are any harmonic functions, then

$$\left.\begin{aligned}\nabla^2(x\psi_1)&=2\frac{\partial\psi_1}{\partial x},\\ \nabla^2(y\psi_2)&=2\frac{\partial\psi_2}{\partial y},\\ \nabla^2(z\psi_3)&=2\frac{\partial\psi_3}{\partial z}.\end{aligned}\right\}\qquad(9.34)$$

On the basis of these identities we can easily prove the following very important theorem. If $\varphi(x, y, z)$ is a harmonic function, then

$$\Phi=x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}\qquad(9.35)$$

is likewise a harmonic function.

Since the derivatives of the function φ will also be harmonic functions, we have the right to set in (9.34)

$$\psi_1=\frac{\partial\varphi}{\partial x};\quad \psi_2=\frac{\partial\varphi}{\partial y};\quad \psi_3=\frac{\partial\varphi}{\partial z}.$$

Next, adding up identities (9.34) by members, we find

$$\nabla^2\left(x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}\right)=2\nabla^2\varphi=0,$$

which was to be proved.

The function Φ can be given a simple vector representation. In fact, if r is the radius-vector of a point (x, y, z), then

$$\left.\begin{aligned}x&=r\cos(r, x),\\ y&=r\cos(r, y),\\ z&=r\cos(r, z).\end{aligned}\right\}\qquad(9.36)$$

Further, if $\nabla\varphi$ is the gradient of the function φ or its derivative along the normal $|\nabla\varphi|=\frac{d\varphi}{dn}$, then

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial x}&=\frac{d\varphi}{dn}\cos\left(\frac{d\varphi}{dn}, x\right),\\ \frac{\partial\varphi}{\partial y}&=\frac{d\varphi}{dn}\cos\left(\frac{d\varphi}{dn}, y\right),\\ \frac{\partial\varphi}{\partial z}&=\frac{d\varphi}{dn}\cos\left(\frac{\partial\varphi}{dn}, z\right).\end{aligned}\right\}\qquad(9.37)$$

Substituting from (9.36) and (9.37) in (9.35), we get

$$\Phi=\frac{d\varphi}{dn}r\left[\cos\left(\frac{d\varphi}{dn}, x\right)\cos(r, x)+\right.$$
$$\left.+\cos\left(\frac{d\varphi}{dn}, y\right)\cos(r, y)+\cos\left(\frac{d\varphi}{dn}, z\right)\cos(r, z)\right],$$

or

$$\Phi=\frac{d\varphi}{dn}r\cos\left(\frac{d\varphi}{dn},\ r\right). \tag{9.38}$$

But

$$\frac{d\varphi}{dn}\cos\left(\frac{d\varphi}{dn},\ r\right)=\frac{d\varphi}{dr}\text{[1]};$$

therefore, we obtain from (9.38) the following vector expression of the function:

$$\Phi=x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}=r\frac{\partial\varphi}{\partial r}. \tag{9.39}$$

59. BIHARMONIC EQUATION

A differential equation of the form

$$\nabla^2\nabla^2 f(x,\ y,\ z)=0 \tag{9.40}$$

is sometimes called biharmonic.

A function f which satisfies this equation is called biharmonic. Equations of the type (9.40) play an important role in the theory of elasticity. We have already encountered in the plane problem the case where the function f depends only on two variables (x, y); it has been shown there that the stress function must satisfy the biharmonic equation (see Sec. 39)

$$\nabla^2\nabla^2\varphi(x,\ y)=0.$$

Later we shall show that Lamé's equations (VI) of Sec. 25 in the absence of body forces $(X=Y=Z=0)$ and Beltrami's equations (VII) of Sec. 36 can also be reduced to biharmonic equations. Bearing this in mind, we shall now indicate some types of solutions of the biharmonic equation, which are necessary for applications. These solutions were obtained by J. Boussinesq; they are closely connected with harmonic functions, as may quite naturally be expected if we take into consideration a very close relation between harmonic equation (9.22) and biharmonic equation (9.40).

We first note that each harmonic function $\varphi(x, y, z)$ is a solution of the biharmonic equation as it satisfies the equation $\nabla^2\varphi=0$ and, therefore, it certainly satisfies equation (9.40): $\nabla^2\nabla^2\varphi=\nabla^2 0=0$. We thus establish that the first type of solutions of equation (9.40) is represented by any harmonic function.

[1] This follows from the definition of the derivative of a function of a point along a given direction.

Let us now show that if ψ_1, ψ_2, ψ_3 are some harmonic functions, the functions

$$x\psi_1, \ y\psi_2, \ z\psi_3 \tag{9.41}$$

will be solutions of the biharmonic equation. This can easily be done on the basis of the deduction summarised in formulas (9.34). Indeed, by applying the Laplacian operator to both members of the first of equations (9.34), we obtain

$$\nabla^2\nabla^2(x\psi_1) = 2\nabla^2\left(\frac{\partial\psi_1}{\partial x}\right).$$

But since ψ_1 is a harmonic function, $\frac{\partial\psi_1}{\partial x}$ will likewise be a harmonic function; consequently, the right-hand member of the last equality is equal to zero, and hence

$$\nabla^2\nabla^2(x\psi_1) = 0, \tag{9.42}$$

which was to be proved.

By using in the same way the second and third of equalities (9.34), we can show that the remainder of functions (9.41) are also solutions of biharmonic equation (9.40). We thus arrive at the second type of solutions of the biharmonic equation in the form of functions (9.41).

If we have found two solutions f_1 and f_2 of equation (9.40), their sum likewise is a solution of this equation. Indeed, it can easily be verified that

$$\nabla^2(f_1+f_2) = \nabla^2 f_1 + \nabla^2 f_2.$$

By applying once again Laplace's operator to both members of this equality, we obtain

$$\nabla^2\nabla^2(f_1+f_2) = \nabla^2\nabla^2 f_1 + \nabla^2\nabla^2 f_2;$$

but the right-hand member of this equation is zero, since f_1 and f_2 are solutions of equation (9.40). It follows that

$$\nabla^2\nabla^2(f_1+f_2) = 0,$$

which was to be proved. By the use of this conclusion we obtain the third type of solutions of the biharmonic equation by summing up the solutions of the first and second type:

$$\left.\begin{aligned} F_1 &= \varphi_1 + x\psi_1, \\ F_2 &= \varphi_2 + y\psi_2, \\ F_3 &= \varphi_3 + z\psi_3, \end{aligned}\right\} \tag{9.43}$$

where φ_1, φ_2, φ_3, ψ_1, ψ_2, ψ_3 are any harmonic functions. Instead of these functions, one can, of course, take any of their partial

derivatives. Later we shall need, for instance, three solutions of the biharmonic equation of the following form [of the type of the third of equations (9.43)]:

$$\left.\begin{aligned} F_1 &= \varphi_1 + z\frac{\partial\psi}{\partial x}, \\ F_2 &= \varphi_2 + z\frac{\partial\psi}{\partial y}, \\ F_3 &= \varphi_3 + z\frac{\partial\psi}{\partial z}, \end{aligned}\right\} \tag{9.44}$$

where φ_1, φ_2, φ_3 and ψ are harmonic functions.

Let us now show that a function of the form

$$r^2\psi \tag{9.45}$$

is a solution of the biharmonic equation if ψ is a harmonic function. For this purpose let us set up the Laplacian operator of function (9.45); we get

$$\frac{\partial(r^2\psi)}{\partial x} = 2r\frac{\partial r}{\partial x}\psi + r^2\frac{\partial\psi}{\partial x},$$

but from (9.25) we have

$$2r\frac{\partial r}{\partial x} = 2x, \tag{9.46}$$

consequently,

$$\frac{\partial(r^2\psi)}{\partial x} = 2x\psi + r^2\frac{\partial\psi}{\partial x}.$$

Further,

$$\frac{\partial^2(r^2\psi)}{\partial x^2} = 2\psi + 2x\frac{\partial\psi}{\partial x} + 2r\frac{\partial r}{\partial x}\frac{\partial\psi}{\partial x} + r^2\frac{\partial^2\psi}{\partial x^2},$$

or, by using (9.46),

$$\frac{\partial^2(r^2\psi)}{\partial x^2} = 2\psi + 4x\frac{\partial\psi}{\partial x} + r^2\frac{\partial^2\psi}{\partial x^2}.$$

Similarly, we get

$$\frac{\partial^2(r^2\psi)}{\partial y^2} = 2\psi + 4y\frac{\partial\psi}{\partial y} + r^2\frac{\partial^2\psi}{\partial y^2},$$

$$\frac{\partial^2(r^2\psi)}{\partial z^2} = 2\psi + 4z\frac{\partial\psi}{\partial z} + r^2\frac{\partial^2\psi}{\partial z^2}.$$

By adding up the last three equalities by members, we find

$$\nabla^2(r^2\psi) = 6\psi + 4\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y} + z\frac{\partial\psi}{\partial z}\right) + r^2\nabla^2\psi;$$

as ψ is assumed to be a harmonic function, the last term vanishes, and we obtain

$$\nabla^2(r^2\psi)=6\psi+4\left(x\frac{\partial\psi}{\partial x}+y\frac{\partial\psi}{\partial y}+z\frac{\partial\psi}{\partial z}\right). \qquad (9.47)$$

But the right-hand member of this equation is the sum of two solutions of the harmonic equation [cf. formula (9.35)]; therefore, it too is a harmonic function. Consequently, the left-hand member $\nabla^2(r^2\psi)$ also represents a harmonic function. Therefore

$$\nabla^2\nabla^2(r^2\psi)=0; \qquad (9.48)$$

hence, function (9.45) is in fact a solution of the biharmonic equation. This is a new, the fourth, type of solutions. As

$$\frac{\partial\psi}{\partial x},\quad \frac{\partial\psi}{\partial y},\quad \frac{\partial\psi}{\partial z}$$

are harmonic functions, the functions

$$r^2\frac{\partial\psi}{\partial x},\quad r^2\frac{\partial\psi}{\partial y},\quad r^2\frac{\partial\psi}{\partial z}$$

will likewise be solutions of the biharmonic equation. By summing up these solutions with the solutions of the first type φ_1, φ_2, φ_3, we obtain the fifth type of solutions:

$$\left.\begin{aligned} F_1&=\varphi_1+r^2\frac{\partial\psi}{\partial x},\\ F_2&=\varphi_2+r^2\frac{\partial\psi}{\partial y},\\ F_3&=\varphi_3+r^2\frac{\partial\psi}{\partial z}.\end{aligned}\right\} \qquad (9.49)$$

If we add to the right-hand members solutions of the first type in the form of functions

$$-a^2\frac{\partial\psi}{\partial x},\quad -a^2\frac{\partial\psi}{\partial y},\quad -a^2\frac{\partial\psi}{\partial z}$$

where a is a constant number, we obtain solutions of the following, the sixth, type:

$$\left.\begin{aligned} F_1&=\varphi_1+(r^2-a^2)\frac{\partial\psi}{\partial x},\\ F_2&=\varphi_2+(r^2-a^2)\frac{\partial\psi}{\partial y},\\ F_3&=\varphi_3+(r^2-a^2)\frac{\partial\psi}{\partial z}.\end{aligned}\right\} \qquad (9.50)$$

60. REDUCTION OF LAMÉ'S AND BELTRAMI'S EQUATIONS TO BIHARMONIC EQUATIONS

The foregoing particular solutions of biharmonic equations of the types (9.44) and (9.49) are of great importance in the solution of the elasticity problem in terms of displacements since Lamé's equations, the basic equations involved in this method,

$$\left.\begin{aligned} (\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\nabla^2 u+X\rho=0,\\ (\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\nabla^2 v+Y\rho=0,\\ (\lambda+\mu)\frac{\partial\theta}{\partial z}+\mu\nabla^2 w+Z\rho=0 \end{aligned}\right\} \tag{VI}$$

in the absence of body forces, i.e., when $X=Y=Z=0$, can easily be reduced to three independent biharmonic equations. This circumstance considerably simplifies the solution of the problem.

Omitting body forces, we write equations (VI) in the form (VI′) of Sec. 25:

$$\left.\begin{aligned} \nabla^2 u+(k+1)\frac{\partial\theta}{\partial x}=0,\\ \nabla^2 v+(k+1)\frac{\partial\theta}{\partial y}=0,\\ \nabla^2 w+(k+1)\frac{\partial\theta}{\partial z}=0, \end{aligned}\right\} \tag{9.51}$$

where, according to (3.17),

$$k=\frac{2\sigma}{1-2\sigma}.$$

Accordingly, the equations of Hooke's law are written in the form (3.18)

$$\left.\begin{aligned} X_x&=\mu(2e_{xx}+k\theta),\\ Y_y&=\mu(2e_{yy}+k\theta),\\ Z_z&=\mu(2e_{zz}+k\theta),\\ Y_z&=\mu e_{yz},\\ Z_x&=\mu e_{zx},\\ X_y&=\mu e_{xy}. \end{aligned}\right\} \tag{9.52}$$

Thus, equations (9.51) and (9.52) are expressed in terms of two elastic constants μ and k.

Turning now to Lamé's equations (9.51), we recall that [Sec. 36, equation (5.55)] in our case, when there are no body forces, the dilatational strain θ satisfies Laplace's equation, i.e., is a har-

monic function

$$\nabla^2\theta = 0.$$

The derivatives

$$\frac{\partial\theta}{\partial x}, \quad \frac{\partial\theta}{\partial y}, \quad \frac{\partial\theta}{\partial z}$$

will likewise be harmonic functions; therefore

$$\nabla^2 \frac{\partial\theta}{\partial x} = 0; \quad \nabla^2 \frac{\partial\theta}{\partial y} = 0; \quad \nabla^2 \frac{\partial\theta}{\partial z} = 0.$$

On this basis, applying Laplace's operator to equations (9.51), we observe that they reduce to the form

$$\left.\begin{aligned} \nabla^2\nabla^2 u &= 0, \\ \nabla^2\nabla^2 v &= 0, \\ \nabla^2\nabla^2 w &= 0. \end{aligned}\right\} \tag{9.53}$$

Of particular importance is the fact that these equations are independent, i.e., only one of the unknown functions u, v, w enters in each of them, and since these equations are biharmonic, the conclusions of Sec. 59 will enable us to find without difficulty any number of their particular solutions [1].

Let us now turn to Beltrami's equations [(VII) of Sec. 36]. We had [see formulas (5.55) of Sec. 36 and (3.14) of Sec. 18]: $\nabla^2\theta = 0$, $\Theta = (3\lambda + 2\mu)\theta$. Hence we obtain $\nabla^2\Theta = 0$, i.e., in the absence of body forces the function $\Theta = X_x + Y_y + Z_z$ is harmonic.

We also take into account that, for instance,

$$\nabla^2\left(\frac{\partial^2\Theta}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}(\nabla^2\Theta) = 0;$$

then, by applying Laplace's operator to each of Beltrami's equations (VII), we obtain

$$\left.\begin{aligned} \nabla^2\nabla^2 X_x &= 0; & \nabla^2\nabla^2 Y_z &= 0; \\ \nabla^2\nabla^2 Y_y &= 0; & \nabla^2\nabla^2 Z_x &= 0; \\ \nabla^2\nabla^2 Z_z &= 0; & \nabla^2\nabla^2 X_y &= 0. \end{aligned}\right\} \tag{9.54}$$

Hence, in the absence of body forces, Lamé's and Beltrami's equations reduce to biharmonic equations. A very important conclusion follows therefrom: all the basic unknown functions in the general problem of the theory of elasticity

$$u, \quad v, \quad w;$$

$$X_x, \quad Y_y, \quad Z_z, \quad Y_z, \quad Z_x, \quad X_y$$

are biharmonic functions.

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[1] Though, as we shall see later, not all of these solutions will satisfy the initial equations (9.51).

61. BOUSSINESQ'S METHOD; APPLICATION OF HARMONIC FUNCTIONS TO SEEKING OF PARTICULAR SOLUTIONS OF LAMÉ'S EQUATIONS

Considerations of Sections 59 and 60 show that solutions of Lamé's equations can be sought as biharmonic functions by assigning their expressions in the form of (9.44), (9.49) and (9.50); the harmonic functions φ_1, φ_2, φ_3, ψ involved in them should be determined so as to satisfy Lamé's equations and prescribed boundary conditions.

We shall investigate solutions of the type (9.44):

$$\left.\begin{aligned} u &= \varphi_1 + z\frac{\partial\psi}{\partial x}, \\ v &= \varphi_2 + z\frac{\partial\psi}{\partial y}, \\ w &= \varphi_3 + z\frac{\partial\psi}{\partial z}, \end{aligned}\right\} \tag{9.55}$$

and solutions of the type (9.49):

$$\left.\begin{aligned} u &= \varphi_1 + r^2\frac{\partial\psi}{\partial x}, \\ v &= \varphi_2 + r^2\frac{\partial\psi}{\partial y}, \\ w &= \varphi_3 + r^2\frac{\partial\psi}{\partial z}. \end{aligned}\right\} \tag{9.56}$$

We shall not use solutions of the type (9.50) here and shall only indicate that they are needed for the problem of equilibrium of an elastic sphere. We should note, however, the following: instead of Lamé's equations (9.51), we have obtained, by differentiation, equations (9.53) of higher order but simpler form and found the solutions of these equations (9.55) and (9.56). One cannot say, however, that (9.55) and (9.56) will necessarily be solutions of Lamé's equations (9.51): the solutions of Lamé's equations will always satisfy equations (9.53), since the latter are a consequence of Lamé's equations; but because of the higher order of equations (9.53), they will have a wider class of solutions than equations (9.51). We shall thus have to ascertain in what conditions (9.55) and (9.56) will satisfy equations (9.51); to this end it is obviously necessary to substitute (9.55) and (9.56) in (9.51) [1].

[1] It will be noted that this procedure has been applied more than once in Sections 32, 33, and 34.

Let us examine solution (9.55). For substitution in (9.51) we calculate first the functions

$$\theta; \quad \nabla^2 u; \quad \nabla^2 v; \quad \nabla^2 w.$$

We have

$$\frac{\partial u}{\partial x}=\frac{\partial \varphi_1}{\partial x}+z\frac{\partial^2 \psi}{\partial x^2},$$

$$\frac{\partial v}{\partial y}=\frac{\partial \varphi_2}{\partial y}+z\frac{\partial^2 \psi}{\partial y^2},$$

$$\frac{\partial w}{\partial z}=\frac{\partial \varphi_3}{\partial z}+z\frac{\partial^2 \psi}{\partial z^2}+\frac{\partial \psi}{\partial z}.$$

Adding up these equations by members and taking into account that the function ψ is harmonic, we find

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}+\frac{\partial \psi}{\partial z}. \tag{9.57}$$

Further we obtain

$$\nabla^2 u=\nabla^2\varphi_1+\nabla^2\left(z\frac{\partial \psi}{\partial x}\right);$$

as the functions φ_1 and ψ are harmonic, we have

$$\nabla^2\varphi_1=0$$

and, according to (9.34),

$$\nabla^2\left(z\frac{\partial \psi}{\partial x}\right)=2\frac{\partial^2 \varphi}{\partial x\,\partial z}=2\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial z}\right),$$

i.e.,

$$\nabla^2 u=2\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial z}\right),$$

in exactly the same manner we find

$$\nabla^2 v=2\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial z}\right),$$

$$\nabla^2 w=2\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right). \tag{9.58}$$

The results (9.57) and (9.58) are introduced in equations (9.51):

$$\frac{\partial}{\partial x}\left(2\frac{\partial \psi}{\partial z}\right)+(k+1)\frac{\partial}{\partial x}\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}+\frac{\partial \psi}{\partial z}\right)=0$$

or

$$\frac{\partial}{\partial x}\left[2\frac{\partial \psi}{\partial z}+(k+1)\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}+\frac{\partial \psi}{\partial z}\right)\right]=0.$$

By making reduction in the square brackets, we get

$$\frac{\partial}{\partial x}\left[(k+1)\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}\right)+(k+3)\frac{\partial \psi}{\partial z}\right]=0.$$

The same computations for the second and third equations yield

$$\frac{\partial}{\partial y}\left[(k+1)\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}\right)+(k+3)\frac{\partial \psi}{\partial z}\right]=0,$$

$$\frac{\partial}{\partial z}\left[(k+1)\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}\right)+(k+3)\frac{\partial \psi}{\partial z}\right]=0.$$

These three equations are satisfied only if

$$(k+1)\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}\right)+(k+3)\frac{\partial \psi}{\partial z}=C, \qquad (9.59)$$

where C is an arbitrary constant.

Equation (9.59) represents just the required condition in which functions (9.55) will be a solution of the system of Lamé's equations (9.51). The arbitrary constant C in (9.59) is of no importance and we set it equal to zero [1]. Hence

$$\frac{\partial \psi}{\partial z}=-\frac{k+1}{k+3}\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}\right). \qquad (9.60)$$

We come to the following conclusion: a particular solution of the system of Lamé's equations

$$\nabla^2 u+(k+1)\frac{\partial \theta}{\partial x}=0,$$

$$\nabla^2 v+(k+1)\frac{\partial \theta}{\partial y}=0,$$

$$\nabla^2 w+(k+1)\frac{\partial \theta}{\partial z}=0$$

can be obtained in the form

$$\left.\begin{aligned} u&=\varphi_1+z\frac{\partial \psi}{\partial x},\\ v&=\varphi_2+z\frac{\partial \psi}{\partial y},\\ w&=\varphi_3+z\frac{\partial \psi}{\partial z},\end{aligned}\right\} \qquad (9.61)$$

where φ_1, φ_2, φ_3 are arbitrary harmonic functions while the harmonic function ψ is determined from equation (9.60). In this way ψ one can obviously obtain a great number of particular solutions, there being available the arbitrariness of the choice of harmonic functions φ_1, φ_2, φ_3.

[1] Indeed, without violating the generality of equation (9.59), we can set $C=D(k+3)$, where D is a new arbitrary constant; we denote $\psi-Dz=\psi_1$ will obviously be a harmonic function, just as ψ; then (9.59) becomes

$$(k+1)\left(\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}\right)+(k+3)\frac{\partial \psi_1}{\partial z}=0.$$

Let us now consider the form (9.56) of the solution of biharmonic equations (9.53) and establish in what conditions this solution satisfies Lamé's equations (9.51). We set up, on the basis of (9.56), the expression of the volume expansion θ. We obtain

$$\frac{\partial u}{\partial x}=\frac{\partial \varphi_1}{\partial x}+\frac{\partial}{\partial x}\left(r^2\frac{\partial \psi}{\partial x}\right); \tag{9.62}$$

but

$$\frac{\partial}{\partial x}\left(r^2\frac{\partial \psi}{\partial x}\right)=\frac{\partial (r^2)}{\partial x}\frac{\partial \psi}{\partial x}+r^2\frac{\partial^2 \psi}{\partial x^2},$$

or, on the basis of (9.25),

$$\frac{\partial}{\partial x}\left(r^2\frac{\partial \psi}{\partial x}\right)=2x\frac{\partial \psi}{\partial x}+r^2\frac{\partial^2 \psi}{\partial x^2}.$$

We introduce this in (9.62); in a similar manner we calculate the remaining derivatives required for θ:

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial \varphi_1}{\partial x}+2x\frac{\partial \psi}{\partial x}+r^2\frac{\partial^2 \psi}{\partial x^2},\\ \frac{\partial v}{\partial y}&=\frac{\partial \varphi_2}{\partial y}+2y\frac{\partial \psi}{\partial y}+r^2\frac{\partial^2 \psi}{\partial y^2},\\ \frac{\partial w}{\partial z}&=\frac{\partial \varphi_3}{\partial z}+2z\frac{\partial \psi}{\partial z}+r^2\frac{\partial^2 \psi}{\partial z^2}.\end{aligned}\right\} \tag{9.63}$$

By adding up these equalities by members and remembering that

$$\nabla^2\psi=\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}=0,$$

we find

$$\theta=\theta_0+2\Phi, \tag{9.64}$$

where the following abbreviated notations are introduced [1]:

$$\theta_0=\frac{\partial \varphi_1}{\partial x}+\frac{\partial \varphi_2}{\partial y}+\frac{\partial \varphi_3}{\partial z}, \tag{9.65}$$

$$\Phi=x\frac{\partial \psi}{\partial x}+y\frac{\partial \psi}{\partial y}+z\frac{\partial \psi}{\partial z}. \tag{9.66}$$

We now proceed to set up Laplace's operators:

$$\nabla^2 u,\quad \nabla^2 v,\quad \nabla^2 w.$$

The first of equations (9.56) yields

$$\nabla^2 u=\nabla^2\left(r^2\frac{\partial \psi}{\partial x}\right), \tag{9.67}$$

since $\nabla^2\varphi_1=0$.

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[1] See formula (9.39).

To write out the right-hand member of (9.67) we make use of formula (9.47), replacing in it ψ by $\frac{\partial\psi}{\partial x}$:

$$\nabla^2 u = 6\left(\frac{\partial\psi}{\partial x} + 4\left(x\frac{\partial^2\psi}{\partial x^2} + y\frac{\partial^2\psi}{\partial x\,\partial y} + z\frac{\partial^2\psi}{\partial x\,\partial z}\right)\right). \tag{9.68}$$

We transform the bracketed expression in the right-hand member on the basis of the following considerations:

$$x\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) - \frac{\partial\psi}{\partial x},$$

$$y\frac{\partial^2\psi}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(y\frac{\partial\psi}{\partial y}\right),$$

$$z\frac{\partial^2\psi}{\partial x\,\partial z} = \frac{\partial}{\partial x}\left(z\frac{\partial\psi}{\partial z}\right).$$

Adding up these identities by members and using notation (9.66), we find

$$x\frac{\partial^2\psi}{\partial x^2} + y\frac{\partial^2\psi}{\partial x\,\partial y} + z\frac{\partial^2\psi}{\partial x\,\partial z} = \frac{\partial\Phi}{\partial x} - \frac{\partial\psi}{\partial x}.$$

Introducing this in (9.68), we get

$$\nabla^2 u = 2\frac{\partial\psi}{\partial x} + 4\frac{\partial\Phi}{\partial x} = 2\frac{\partial}{\partial x}(\psi + 2\Phi). \tag{9.69}$$

In exactly the same way we obtain

$$\left.\begin{aligned} \nabla^2 v &= 2\frac{\partial}{\partial y}(\psi + 2\Phi), \\ \nabla^2 w &= 2\frac{\partial}{\partial z}(\psi + 2\Phi). \end{aligned}\right\} \tag{9.69}$$

We substitute (9.64) and (9.69) in the first of equations (9.51):

$$2\frac{\partial}{\partial x}(\psi + 2\Phi) + (k+1)\frac{\partial}{\partial x}(\theta_0 + 2\Phi) = 0,$$

or

$$\frac{\partial}{\partial x}[(k+1)\,\theta_0 + 2\psi + 2(k+3)\,\Phi] = 0.$$

The other two of equations (9.51) yield:

$$\frac{\partial}{\partial y}[(k+1)\,\theta_0 + 2\psi + 2(k+3)\,\Phi] = 0,$$

$$\frac{\partial}{\partial z}[(k+1)\,\theta_0 + 2\psi + 2(k+3)\,\Phi] = 0.$$

Hence we conclude

$$(k+1)\,\theta_0 + 2\psi + 2(k+3)\,\Phi = \text{const.}$$

Without loss in generality of this equation we can set its constant right-hand member equal to zero[1]; dividing the equation through by $2(k+3)$, we find

$$\Phi+\frac{1}{k+3}\psi=-\frac{k+1}{2(k+3)}\theta_0. \tag{9.70}$$

Recalling now that according to formula (9.39) of Sec. 58 the function Φ has the vector expression

$$\Phi=r\frac{\partial\psi}{\partial r},$$

equation (9.70) is rewritten as

$$r\frac{\partial\psi}{\partial r}+\frac{1}{k+3}\psi=-\frac{k+1}{2(k+3)}\theta_0,$$

or

$$r\frac{\partial\psi}{\partial r}+\alpha\psi=-\beta\theta_0, \tag{9.71}$$

where

$$\alpha=\frac{1}{k+3};\quad \beta=\frac{k+1}{2(k+3)}.$$

If we pass from the Cartesian co-ordinates x, y, z to another system in which one of the co-ordinates is the radius-vector r (for instance, to polar co-ordinates in space), the derivative $\frac{\partial\psi}{\partial r}$ of the function ψ along the direction r may be treated as the partial derivative of ψ with respect to the variable r; obviously, the other two co-ordinates determining the direction of the radius-vector r are assumed to remain constant. If the same reasoning is applied to the function θ_0, equation (9.71) may be looked upon as an ordinary linear differential equation with respect to the function ψ. By integrating it, we find ψ as a function of the radius-vector r; however, the arbitrary constant, which will be involved here, will in general depend on the other two co-ordinates, i.e., it will depend on the direction of the radius-vector r.

Equation (9.71) is integrated by the usual method of substitution:

$$\psi=UV, \tag{9.72}$$

where U and V are two new functions of r. Substituting this in (9.71), we find

$$r(UV'+U'V)+\alpha UV=-\beta\theta_0,$$

or

$$rUV'+(rU'+\alpha U)V=-\beta\theta_0. \tag{9.73}$$

[1] This constant can be incorporated in either of functions θ_0 or ψ in the left-hand member of the equation.

We select the function U from the condition

$$rU' + \alpha U = 0 \quad \text{or} \quad r\frac{\partial U}{\partial r} = -\alpha U. \tag{9.74}$$

Hence

$$\frac{\partial U}{U} = -\alpha \frac{dr}{r}.$$

Integration yields

$$\ln U = -\alpha \ln r \quad \text{or} \quad U = r^{-\alpha}. \tag{9.75}$$

On the basis of (9.74) and (9.75) equation (9.73) takes the form

$$r^{-\alpha+1}\frac{dV}{dr} = -\beta\theta_0,$$

or

$$\frac{dV}{dr} = -\beta r^{\alpha-1}\theta_0.$$

Multiplying both members by dr and integrating, we find

$$V = -\beta \int \theta_0 r^{\alpha-1}\,dr + C.$$

Substituting from this and from (9.75) in (9.72), we find the function ψ:

$$\psi = -\beta r^{-\alpha}\int \theta_0 r^{\alpha-1}\,dr + Cr^{-\alpha}. \tag{9.76}$$

Here, as pointed out above, the constant C depends in general on the direction of the radius-vector r, i.e., when C is equal to a constant number, (9.76) gives a law of variation of the function ψ with a motion in the chosen direction drawn from the origin.

62. EFFECT OF A LOAD ON A MEDIUM BOUNDED BY A PLANE (BOUSSINESQ'S PROBLEM)

A similar problem was treated in Sec. 49 for the case of plane strain; we assumed there that the load was distributed uniformly along an infinite straight line in the plane bounding the medium. Here we shall indicate a method of solving the problem in the general case when the medium under consideration is acted upon by an arbitrary load; the medium itself will be called the half-space for brevity, and the plane which bounds it—the boundary.

The origin of co-ordinates is chosen somewhere on the boundary; the axes Ox and Oy are directed in the boundary plane and the axis Oz normally to it along the outward normal. Then the

problem in question can be formulated as follows: it is required to find the stresses and displacements at any point of the half-space (i. e., for any x, y and z less than zero), if the stresses due to the load X_z, Y_z, Z_z are prescribed on the boundary $z=0$ as functions of x and y. If the displacements u, v, w, rather than the stresses, were prescribed on the boundary, the solution of the problem could be found directly in the form (9.61). Indeed, if it is specified that for $z=0$ on the boundary

$$u=f_1(x, y); \quad v=f_2(x, y); \quad w=f_3(x, y),$$

then, putting $z=0$ in (9.61), we find that

$$\varphi_1(x, y, 0)=f_1(x, y); \ \varphi_2(x, y, 0)=f_2(x, y); \ \varphi_3(x, y, 0)=f_3(x, y).$$

Hence, the harmonic functions φ_1, φ_2, φ_3 are to be found from the condition that they turn into preassigned functions f_1, f_2, f_3 on the boundary $z=0$; this is the basic boundary value problem of the theory of potential, which admits quite a definite solution; after determining the functions φ_1, φ_2 and φ_3, we find the function ψ by (9.60) and, introducing all this in (9.61), we complete the solution of the problem.

However, it is the stresses that we consider to be given on the boundary and, therefore, the procedure of solving the problem will be somewhat complicated. Let the stresses on the boundary be given in the form of functions of x and y as follows:

$$Z_z=\Phi_1(x, y); \quad Z_x=\Phi_2(x, y); \quad Z_y=\Phi_3(x, y). \tag{9.77}$$

Since we are solving the problem in terms of displacements (Chapter IV, Sec. 24), we express the left-hand members of these equalities in terms of displacements u, v, w by using Hooke's law (9.52) and Cauchy's equations (III) of Sec. 24:

$$\left.\begin{aligned}
\mu\left(2\frac{\partial w}{\partial z}+k\theta\right)&=\Phi_1,\\
\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)&=\Phi_2,\\
\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)&=\Phi_3.
\end{aligned}\right\} \tag{9.78}$$

These equations are valid only on the boundary, i.e., for $z=0$. Let us now assume that we have found three harmonic functions:

$$\omega_1(x, y, z); \quad \omega_2(x, y, z); \quad \omega_3(x, y, z)$$

such that on the boundary $z=0$ they turn into prescribed functions (9.77), apart from a constant factor μ which we introduce

to simplify subsequent computations:

$$\left.\begin{aligned}\mu\omega_1(x,\ y,\ 0) &= \Phi_1(x,\ y),\\ \mu\omega_2(x,\ y,\ 0) &= \Phi_2(x,\ y),\\ \mu\omega_3(x,\ y,\ 0) &= \Phi_3(x,\ y).\end{aligned}\right\} \tag{9.79}$$

We substitute these values of Φ_1, Φ_2 and Φ_3 in the right-hand members of equations (9.78) and replace u, v and w in the left-hand members by their expressions from (9.61) and (9.57); we obtain

$$2\frac{\partial\varphi_3}{\partial z}+2z\frac{\partial^2\psi}{\partial z^2}+2\frac{\partial\psi}{\partial z}+k\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial z}+\frac{\partial\psi}{\partial z}\right)=\omega_1,$$

$$\frac{\partial\varphi_1}{\partial z}+z\frac{\partial^2\psi}{\partial z\,\partial x}+\frac{\partial\psi}{\partial x}+\frac{\partial\varphi_3}{\partial x}+z\frac{\partial^2\psi}{\partial z\,\partial x}=\omega_2,$$

$$\frac{\partial\varphi_3}{\partial y}+z\frac{\partial^2\psi}{\partial y\,\partial z}+\frac{\partial\varphi_2}{\partial z}+z\frac{\partial^2\psi}{\partial y\,\partial z}+\frac{\partial\psi}{\partial y}=\omega_3.$$

In these equations we omit the terms involving the factor z (as $z=0$ on the boundary) and note that on the basis of (9.57) and (9.60)

$$\theta=\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial z}+\frac{\partial\psi}{\partial z}=-\frac{k+3}{k+1}\frac{\partial\psi}{\partial z}+\frac{\partial\psi}{\partial z}=-\frac{2}{k+1}\frac{\partial\psi}{\partial z}. \tag{9.80}$$

Then the foregoing equations become

$$\left.\begin{aligned}2\frac{\partial\varphi_3}{\partial z}+\frac{2}{k+1}\frac{\partial\psi}{\partial z}&=\omega_1,\\ \frac{\partial\varphi_1}{\partial z}+\frac{\partial\varphi_3}{\partial x}+\frac{\partial\psi}{\partial x}&=\omega_2,\\ \frac{\partial\varphi_2}{\partial z}+\frac{\partial\varphi_3}{\partial y}+\frac{\partial\psi}{\partial y}&=\omega_3.\end{aligned}\right\} \tag{9.81}$$

These equations, as already stated, hold on the boundary of the half-space. Both their left-hand and right-hand members, however, are harmonic functions; it is known that if two harmonic functions coincide on the boundary of a region, they are identical in the whole region [1]. It follows from this that equations (9.81) are valid in the whole half-space $z<0$. This conclusion enables us, by eliminating the three functions φ_1, φ_2 and φ_3 from (9.81) and (9.60), to express ψ in terms of ω_1, ω_2, ω_3 which are assumed to have been determined previously for any point of the half-space.

[1] This theorem is valid if the above functions, besides satisfying Laplace's equation, have no singularities in the region of existence and particularly if they do not become infinite.

We differentiate the first of equations (9.81) with respect to z, the second with respect to x and the third with respect to y and add up the results by members; in the equation thus obtained a number of terms vanish since the functions φ_1, φ_2, φ_3 and ψ are harmonic and therefore

$$\nabla^2\varphi_1 = \nabla^2\varphi_2 = \nabla^2\varphi_3 = \nabla^2\psi = 0.$$

We find the following result:

$$-\frac{k-1}{k+1}\frac{\partial^2\psi}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}\right) = \frac{\partial\omega_1}{\partial z} + \frac{\partial\omega_2}{\partial x} + \frac{\partial\omega_3}{\partial y}.$$

In order to eliminate the functions φ_1, φ_2, φ_3 from this we make use of equation (9.60) and obtain finally

$$\frac{\partial^2\psi}{\partial z^2} = -\frac{1}{2}\left(\frac{\partial\omega_1}{\partial z} + \frac{\partial\omega_2}{\partial x} + \frac{\partial\omega_3}{\partial y}\right). \tag{9.82}$$

After this has been done, the procedure of solving the problem is as follows:

1. According to the prescribed functions Φ_1, Φ_2, Φ_3 [see equations (9.77)], we find harmonic functions ω_1, ω_2, ω_3 satisfying conditions (9.79) on the boundary.
2. By double integration we find the function ψ from (9.82).
3. From (9.81) we seek harmonic functions φ_1, φ_2, φ_3.
4. Substituting the values of ψ and φ in (9.61), we find displacements u, v, w.
5. From Hooke's law (9.52) we find the stresses at an arbitrary point of the half-space.

Subsequently we shall confine ourselves to such cases of loading on the boundary where the resultant is a finite quantity. Then, as z increases beyond all bounds (i.e., for an infinitely removed horizontal section of the half-space), all the displacements and stresses must tend to zero; hence, on the basis of equations (9.61), we note that as $z \to -\infty$ the function ψ and its first derivatives must likewise approach zero. We shall make use of this remark in determining the function ψ from equation (9.82). By integrating it twice with respect to z from $-\infty$ to an arbitrary value of z, we obtain

$$\frac{\partial\psi}{\partial z} = -\int_{-\infty}^{z}\left(\frac{\partial\omega_1}{\partial z} + \frac{\partial\omega_2}{\partial x} + \frac{\partial\omega_3}{\partial y}\right)dz + f_1(x, y),$$

$$\psi = -\frac{1}{2}\int_{-\infty}^{z}\int_{-\infty}^{z}\left(\frac{\partial\omega_1}{\partial z} + \frac{\partial\omega_2}{\partial x} + \frac{\partial\omega_3}{\partial y}\right)dz\,dz + zf_1(x, y) + f_2(x, y).$$

But for $z=-\infty$ the left-hand members of both of these equalities must vanish according to the condition; since the integrals in the right-hand members also vanish, we conclude that

$$f_1=f_2=0,$$

and we finally obtain

$$\psi=-\frac{1}{2}\int_{-\infty}^{z}\int_{-\infty}^{z}\left(\frac{\partial\omega_1}{\partial z}+\frac{\partial\omega_2}{\partial x}+\frac{\partial\omega_3}{\partial y}\right)dz\,dz. \qquad (9.83)$$

The first stage of the foregoing procedure of solving our problem makes it necessary to solve the Dirichlet boundary value problem for harmonic functions ω_1, ω_2, ω_3 [see formulas (9.79)]. As shown by Boussinesq, this problem, however, can easily be solved in the case where the load consists of a single concentrated force applied to any point of the boundary; hence one can pass to the case of an arbitrary load by approximately the same procedure as was employed in Sec. 49 in the corresponding plane problem.

63. EFFECT OF A CONCENTRATED FORCE NORMAL TO THE BOUNDARY AND APPLIED AT THE ORIGIN

The conditions on the boundary of the half-space [see equations (9.77)] in this case are as follows:

1. Functions $\Phi_2=\Phi_3=0$ throughout the boundary.
2. Function Φ_1 is zero throughout the boundary except at the origin ($x=y=0$), where it is infinite.

From these conditions it is necessary to find [cf. conditions (9.79)] harmonic functions ω_1, ω_2, ω_3. But it can be proved that if a harmonic function is equal to zero on the boundary of a region, it is equal to zero everywhere in the region; therefore, from the conditions $\Phi_2=\Phi_3=0$ on the bounrday of the half-space we conclude that $\omega_2=\omega_3=0$ in the whole half-space.

It remains to find function $\omega_1(x, y, z)$ from the condition that it is zero throughout the boundary ($z=0$) except at the origin ($x=y=z=0$), where it has a singularity, for it becomes infinite; the nature of this singularity must be examined more closely to determine better the behaviour of the function on the boundary and in the vicinity of the singular point.

Let us cut out a hemisphere of radius r with centre at the origin O from the half-space; from the condition of equilibrium of this hemisphere it follows (Fig. 93a) that

$$\int Z_\nu dS=P,$$

where Z_v, as usual, denotes the projection of the stress over the area dS of the surface of the hemisphere on the axis Oz; the integral is extended over the surface of the hemisphere. According to the mean value theorem for integrals we obtain

$$\int Z_v\,dS = Z_{vm}\int dS = Z_{vm}\cdot 2\pi r^2,$$

where Z_{vm} is a certain average stress on the surface of the hemisphere. Hence

$$Z_{vm}\cdot 2\pi r^2 = P$$

and, finally,

$$Z_{vm} = \frac{P}{2\pi}\cdot\frac{1}{r^2} = \frac{C}{r^2}. \qquad (9.84)$$

This conclusion refers to a certain average stress on the surface of the hemisphere; from this one can draw, however, an important conclusion that in general the stresses in the half-space due

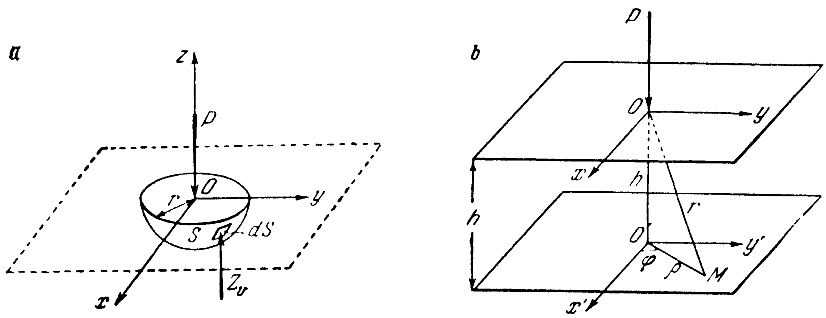

Fig. 93

to the force P, as the point of its application is approached, increase at a rate of the same order as (9.84) where r is the distance from a given point to the point of application of the force.

But, as is seen from (9.77) and (9.79), the function ω_1 is a quantity of the same order as the stress; consequently, it should likewise

(1) vanish everywhere on the boundary except at the origin;

(2) become infinite in the same order as $\frac{1}{r^2}$ when $r \to 0$;

(3) be a harmonic function, i.e., satisfy the equation $\nabla^2\omega_1 = 0$ everywhere except at the origin[1].

[1] In addition, as pointed out above with respect to equations (9.81), it must remain finite everywhere in the half-space, i.e., for $z<0$, including the case of x, y, z tending to infinity.

All these conditions are satisfied by the last of functions (9.31) indicated in Sec. 58; we take it in the form

$$\omega_1 = \frac{Cz}{r^3},$$

where C is an arbitrary constant of integration; later we shall relate it to the magnitude of the force P. Thus, we assume

$$\omega_1 = \frac{Cz}{r^3}; \quad \omega_2 = \omega_3 = 0. \tag{9.85}$$

We proceed to seeking the function ψ; from equation (9.83) we have

$$\psi = -\frac{1}{2} \int_{-\infty}^{z} \int_{-\infty}^{z} \frac{\partial \omega_1}{\partial z}\, dz\, dz = -\frac{1}{2} \int_{-\infty}^{z} \omega_1\, dz.$$

But in Sec. 58 we saw that [cf. (9.27)]

$$\frac{z}{r^3} = \frac{\partial}{\partial z}\left(-\frac{1}{r}\right).$$

Therefore,

$$\psi = -\frac{1}{2} \int_{-\infty}^{z} \frac{\partial}{\partial z}\left(-\frac{C}{r}\right) dz = \frac{C}{2}\frac{1}{r}. \tag{9.86}$$

We calculate the functions φ_1, φ_2, φ_3 from equations (9.81). The first of them gives

$$2\frac{\partial \varphi_3}{\partial z} = \omega_1 - \frac{2}{k+1}\frac{\partial \psi}{\partial z} = \frac{Cz}{r^3} + \frac{2}{k+1}\frac{C}{2}\frac{z}{r^2} = \frac{k+2}{k+1}\frac{Cz}{r^3},$$

but

$$\frac{k+2}{k+1} = 2(1-\sigma).$$

Therefore, we have

$$\left.\begin{aligned} \frac{\partial \varphi_3}{\partial z} &= C(1-\sigma)\frac{z}{r^3}, \\ \varphi_3 &= -\frac{C(1-\sigma)}{r}. \end{aligned}\right\} \tag{9.87}$$

By using the second of equations (9.81), we find φ_1

$$\frac{\partial \varphi_1}{\partial z} = -\frac{\partial \varphi_3}{\partial x} - \frac{\partial \psi}{\partial x} = -\frac{C(1-2\sigma)}{2}\frac{x}{r^3},$$

$$\varphi_1 = -\frac{C(1-2\sigma)x}{2}\int \frac{dz}{r^3}.$$

In taking this integral, we recall that

$$r = \sqrt{z^2+\xi^2}, \quad \text{where} \quad \xi^2 = x^2 + y^2,$$

and apply Euler's substitution

$$r = t + z,$$

where t is a new variable. We have

$$z = -\frac{t^2 - \xi^2}{2t}; \quad dz = -\frac{t^2 + \xi^2}{2t^2}\,dt; \quad r = \frac{t^2 + \xi^2}{2t}.$$

Introducing this in the integral, we obtain

$$\int \frac{dz}{r^3} = -4\int \frac{t\,dt}{(t^2 + \xi^2)^2} = \frac{2}{t^2 + \xi^2} = \frac{1}{r(r - z)},$$

and, therefore,

$$\varphi_1 = -\frac{C(1-2\sigma)}{2}\,\frac{x}{r(r-z)}. \tag{9.88}$$

In the same manner, by using the third of equations (9.81), we find [1]

$$\varphi_2 = -\frac{C(1-2\sigma)}{2}\,\frac{y}{r(r-z)}. \tag{9.89}$$

Now we have only to introduce functions (9.86), (9.87), (9.88) and (9.89) in equations (9.61) and obtain the following expressions of displacements:

$$\left.\begin{aligned} u &= -\frac{C}{2}\left[(1-2\sigma)\frac{x}{r(r-z)} + \frac{xz}{r^3}\right], \\ v &= -\frac{C}{2}\left[(1-2\sigma)\frac{y}{r(r-z)} + \frac{yz}{r^3}\right], \\ w &= -\frac{C}{2}\left[2(1-\sigma)\frac{1}{r} + \frac{z^2}{r^3}\right]. \end{aligned}\right\} \tag{9.90}$$

In addition we shall derive the expression of the volume expansion required in seeking the stresses. For this purpose it is easiest to make use of equation (9.80) by introducing therein the value of the function ψ from (9.86):

$$\theta = -\frac{2}{k+1}\,\frac{\partial\psi}{\partial z} = \frac{C}{k+1}\,\frac{z}{r^3} = C(1-2\sigma)\frac{z}{r^3}. \tag{9.91}$$

With expressions (9.90) and (9.91) we can calculate the stress components by the formulas of Hooke's law (9.52) and Cauchy's equations (III). Omitting the corresponding computations, we

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[1] Here, as in (9.87) and (9.88), in evaluating the indefinite integrals we omit arbitrary constants in view of the fact that functions φ_1, φ_2, φ_3 must vanish as $z \to \infty$.

present the final result:

$$\left.\begin{aligned}
X_x &= C\mu\left\{\frac{3x^2z}{r^5}-(1-2\sigma)\left[\frac{r^2+rz-z^2}{r^3(r-z)}-\frac{x^2(2r-z)}{r^3(r-z)^2}\right]\right\},\\
Y_y &= C\mu\left\{\frac{3y^2z}{r^5}-(1-2\sigma)\left[\frac{r^2+rz-z^2}{r^3(r-z)}-\frac{y^2(2r-z)}{r^3(r-z)^2}\right]\right\},\\
Z_z &= 3C\mu\frac{z^3}{r^5},\\
Y_z &= 3C\mu\frac{yz^2}{r^5},\\
Z_x &= 3C\mu\frac{xz^2}{r^5},\\
X_y &= C\mu\left[\frac{3xyz}{r^5}+(1-2\sigma)\frac{xy(2r-z)}{r^3(r-z)^2}\right].
\end{aligned}\right\} \tag{9.92}$$

It now remains to determine the constant C involved in the expressions of stresses so as to make the load on the boundary (at the origin) reduce to the given force P. For that it is sufficient to require that the resultant of the normal forces $Z_z dS$ (Fig. 93*b*) over any horizontal section of the half-space at a constant depth $z=-h$ be equal to $-P$ (P is the magnitude of the force which is assumed to be compressive). Hence we obtain the condition

$$\int Z_z\,dS=-P,$$

where the integral is extended over the whole plane $z=-h$ parallel to the boundary. Substituting for Z_z its expression (9.92), we find

$$-3C\mu h^3\int\frac{dS}{r^5}=-P. \tag{9.93}$$

To evaluate the integral involved here we make use of the polar co-ordinates ρ and φ (Fig. 93*b*). Let us transfer the integral to these variables. We obtain

$$r=\sqrt{x^2+y^2+h^2}=\sqrt{\rho^2+h^2}.$$

The element of area in polar co-ordinates, as is known, is $dS=\rho\,d\rho\,d\varphi$. Introducing all this in the integral of equality (9.93), we have

$$3C\mu h^3\int\int\frac{\rho\,d\rho\,d\varphi}{(\rho^2+h^2)^{\frac{5}{2}}}=P,$$

or

$$3C\mu h^3\int\limits_0^{\infty}\frac{\rho\,d\rho}{(\rho^2+h^2)^{\frac{5}{2}}}\int\limits_0^{2\pi}d\varphi=P;$$

we carry out integration with respect to φ and then with respect to ρ:

$$\int_0^\infty \frac{\rho\, d\rho}{(\rho^2+h^2)^{\frac{5}{2}}} = \frac{1}{2}\int_0^\infty \frac{d(\rho^2+h^2)}{(\rho^2+h^2)^{\frac{5}{2}}} = \frac{1}{2}\,\frac{(\rho^2+h^2)^{-\frac{3}{2}}}{\left(-\frac{3}{2}\right)}\Bigg|_0^\infty = \frac{1}{3h^3}.$$

The preceding equation thus yields

$$3C\mu h^3 \cdot 2\pi \frac{1}{3h^3} = P,$$

and hence

$$C\mu \cdot 2\pi = P; \quad C = \frac{P}{2\pi\mu}. \tag{9.94}$$

It remains to introduce this value of C in equations (9.92) and (9.90) to obtain the final expressions of stresses and displacements.

As for formulas (9.90) and (9.92), it is necessary to make the reservation we did at the beginning of Sec. 49 for a similar plane problem: expressions (9.90) and (9.92) are valid throughout the half-space except for a small region near the origin (the point of application of the concentrated force) where the stresses are beyond the elastic limit of a given material. In this region Hooke's law, on which the entire derivation is based, does not hold; the applicability of our derivation in the remaining part of the half-space, just as in the problem of Sec. 49, is determined by Saint-Venant's principle.

The foregoing solution of the problem of a concentrated force acting on the boundary of the half-space was based on the expressions of displacements in the form (9.55) subject to condition (9.60).

The problem of a concentrated force acting at a point of an unbounded space is solved in a similar way; for that it is necessary to accept as a starting point the expressions of displacements (9.56) subject to condition (9.76) [1].

A. E. H. Love has shown that Boussinesq's solution (9.90) and (9.92) for a concentrated force can be extended to the case of a distributed load of intensity $q(\xi, \eta)$ acting on the boundary of the half-space, where ξ, η are the co-ordinates of a point in the loaded part of the boundary; we can then assume that the element of area dS at this point is acted upon by the elementary concentrated force

$$q(\xi, \eta)\, dS. \tag{9.95}$$

[1] See E. Trefitz, *Handbuch der Physik*, Band VI. "Mechanik der elastischen Körper", Berlin, 1928.

Summing up the effects of such forces, we obtain the displacements and stresses in the elastic half-space due to the given distributed load. Love's formulas for displacements in this case are given in his treatise on the theory of elasticity, already cited, and also in L. S. Leibenzon, *Theory of Elasticity*, 2nd ed., Moscow, Gostekhizdat, 1947.

It is very easy to obtain the most important formula in applications for the displacement w of a point on the boundary $z=0$ perpendicular to it; for the elementary load (9.95) we have, according to the third of formulas (9.90) and formula (9.94), for $z=0$

$$\delta w = -\frac{q(\xi, \eta)\, dS}{4\pi\mu} 2(1-\sigma)\frac{1}{r'},$$

where

$$r' = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

is the distance from the load (9.95) to the point of the boundary (x, y) at which the displacement is being determined. The expression of δw must be summed up with respect to the coordinates ξ and η for all points of the loaded area; we obtain the expression of the total vertical displacement of the point (x, y) of the boundary:

$$w = -\frac{1-\sigma}{2\pi\mu}\int \frac{q(\xi, \eta)}{r'}\, dS = -\frac{1-\sigma^2}{\pi E}\int \frac{q(\xi, \eta)}{r'}\, d\xi\, d\eta. \quad (9.96)$$

The case of a load distributed on the boundary of the half-space served H. Hertz as a starting point for solving the problem of compression of two bodies bounded by curvilinear surfaces. In Hertz's theory formula (9.96) is fundamental. This same formula underlies many works concerning the theory of analysis of soils as elastic foundations for various kinds of buildings and structures.

We have considered in this section the case of a concentrated force normal to the boundary of the half-space. If the force P is directed in the plane of the base, e.g., along the axis Ox, the whole procedure of solution will be changed only in the sense that in place of (9.85) one must take $\omega_1 = \omega_3 = 0$, $\omega_2 = \frac{Cz}{r^3}$ and carry out all subsequent computations accordingly.

With the solutions for the cases of vertical and horizontal forces one can, of course, obtain displacements and stresses for a force inclined at an arbitrary angle to the boundary.

The foregoing Boussinesq's solutions of the form (9.55) and (9.56) expressed in terms of three arbitrary harmonic functions φ_1, φ_2, φ_3 are not general solutions of the equations of the theory

of elasticity even though they make possible the solution of a great number of problems. The general solution which is expressed, in the absence of body forces, in terms of three arbitrary biharmonic functions was found by Academician B. G. Galerkin [1].

64. SOLUTION OF THE PLANE PROBLEM OF ELASTICITY BY MEANS OF FUNCTIONS OF A COMPLEX VARIABLE

In the preceding sections (58-63) it was ascertained that harmonic functions of three variables x, y, z were of great importance in constructing the solutions of basic equations of the theory of elasticity. Turning to the plane problem we note that we shall work here with harmonic functions in the plane which are very closely related to analytic functions of a complex variable

$$z = x + iy.$$

We shall now carry out sufficiently simple computations of a formal character which will confirm this relation and at the same time prove useful for what follows. We replace the independent variables x and y by other variables z and $\bar{z}$ according to the formulas

$$\left.\begin{aligned} z &= x + iy, \\ \bar{z} &= x - iy. \end{aligned}\right\} \tag{9.97}$$

Obviously, z and $\bar{z}$ are two conjugate complex numbers. By differentiating (9.97) with respect to x and y, we find

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 1; \quad \frac{\partial z}{\partial y} = i; \\ \frac{\partial \bar{z}}{\partial x} &= 1; \quad \frac{\partial \bar{z}}{\partial y} = -i. \end{aligned}\right\} \tag{9.98}$$

Let us consider a function $f(x, y)$ and assume that its independent variables x and y are replaced by the variables z and $\bar{z}$ according to (9.97). Then, in differentiating this function with respect to x and y it is necessary to apply the rule of differentiation of a composite function. For instance,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x},$$

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[1] See, for instance, L. S. Leibenzon, *Theory of Elasticity*, 2nd ed., Moscow, Gostekhizdat, 1947, Chap. IV, Sec. 16.

or, by using (9.98), we obtain

and similarly

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial z} i - \frac{\partial f}{\partial \bar{z}} i. \end{aligned}\right\} \tag{9.99}$$

From (9.99) we immediately find the equalities which will be required later:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} &= 2 \frac{\partial f}{\partial \bar{z}}, \\ \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} &= 2 \frac{\partial f}{\partial z}. \end{aligned}\right\} \tag{9.100}$$

Differentiating (9.99) with respect to x and y respectively and using again (9.98), we get

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} + 2 \frac{\partial^2 f}{\partial z\, \partial \bar{z}} + \frac{\partial^2 f}{\partial \bar{z}^2},$$

$$\frac{\partial^2 f}{\partial y^2} = - \frac{\partial^2 f}{\partial z^2} + 2 \frac{\partial^2 f}{\partial z\, \partial \bar{z}} - \frac{\partial^2 f}{\partial \bar{z}^2}.$$

Adding up these equalities by members, we obtain the formula of transformation of Laplace's operator to the variables z and $\bar{z}$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4 \frac{\partial^2 f}{\partial z\, \partial \bar{z}}. \tag{9.101}$$

Thus Laplace's equation $\nabla^2 f = 0$ in the new co-ordinates becomes

$$\frac{\partial^2 f}{\partial z\, \partial \bar{z}} = 0,$$

and, therefore, we immediately find its general solution in the form

$$f = F_1(z) + F_2(\bar{z}), \tag{9.102}$$

where F_1 and F_2 are arbitrary analytic functions of the variables z and $\bar{z}$ (respectively). The correctness of solution (9.102) is verified by differentiation.

Any harmonic function can thus be represented as the sum of two functions of complex variables z and $\bar{z}$, according to (9.102). This fact underlies several methods of application of a complex variable to the plane problem of the theory of elasticity. The most significant contributions in this respect have been made by G. V. Kolosoff [1] and especially by N. I. Muskhelishvili who devel-

[1] G. V. Kolosoff, ***On One Application of Functions of a Complex Variable to the Plane Problem of Elasticity,*** Jurjev, 1909.

oped Kolosoff's method and worked out a complete theory of this question[1]. Muskhelishvili's investigations and his method have been widely used and inspired a great number of works constituting a special direction in the development of the theory of elasticity during the last decades.

Among earlier investigations we should point out A. E. H. Love's work set forth in his course on the theory of elasticity and also the method proposed by L. N. G. Filon as far back as 1903. Filon showed that by changing the variables considered in the preceding section it is possible to reduce Lamé's equations in the plane problem to the form that lends itself to integration in quadratures, and to find their general solution. He gave, however, no essential applications of the important result obtained and the method was forgotten. We present here Filon's solution because it is obtained by a simple and natural course of reasoning and permits the reader to approach that stage of analysis which opens ways to the effective method of solving the specific problems elaborated by N. I. Muskhelishvili and his school.

65. L. FILON'S METHOD

Consider the case of plane strain in the absence of body forces; Lamé's equations for this case will be obtained from (9.51) of Sec. 60 if we set $w=0$ and regard u and v as functions of x and y only:

$$\left.\begin{aligned} \nabla^2 u+(k+1)\frac{\partial\theta}{\partial x}=0,\\ \nabla^2 v+(k+1)\frac{\partial\theta}{\partial y}=0,\end{aligned}\right\} \tag{9.103}$$

where

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y};\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{9.104}$$

To transfer to new independent variables z and $\bar{z}$ we multiply the second of equations (9.103) by $\pm i$ and add it to the first; we obtain the following two equations:

$$\left.\begin{aligned} \nabla^2(u+iv)+(k+1)\left(\frac{\partial\theta}{\partial x}+i\frac{\partial\theta}{\partial y}\right)=0,\\ \nabla^2(u-iv)+(k+1)\left(\frac{\partial\theta}{\partial x}-i\frac{\partial\theta}{\partial y}\right)=0.\end{aligned}\right\} \tag{9.105}$$

[1] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Groningen, *P.* Noordhoff, 1953.

We now replace the unknown functions $u(x, y)$ and $v(x, y)$ by the new ones U and $\overline{U}$, according to the formulas

$$\left.\begin{aligned} U &= u + iv, \\ \overline{U} &= u - iv. \end{aligned}\right\} \tag{9.106}$$

By using (9.99), we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}},$$

$$\frac{\partial v}{\partial y} = i\frac{\partial v}{\partial z} - i\frac{\partial v}{\partial \bar{z}}.$$

Adding up these equalities by members, we have

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial z}(u + iv) + \frac{\partial}{\partial \bar{z}}(u - iv)$$

or, on the basis of (9.106),

$$\theta = \frac{\partial U}{\partial z} + \frac{\partial \overline{U}}{\partial \bar{z}}. \tag{9.107}$$

Thus θ is expressed in terms of new functions (9.106) and new independent variables z and $\bar{z}$. Furthermore, to transfer (9.105) we have from (9.101) and (9.100)

$$\left.\begin{aligned} \nabla^2(u + iv) &= \nabla^2 U = 4\frac{\partial^2 U}{\partial z\,\partial \bar{z}}, \\ \nabla^2(u - iv) &= \nabla^2 \overline{U} = 4\frac{\partial^2 \overline{U}}{\partial z\,\partial \bar{z}}; \end{aligned}\right\} \tag{9.108}$$

$$\left.\begin{aligned} \frac{\partial \theta}{\partial x} + i\frac{\partial \theta}{\partial y} &= 2\frac{\partial \theta}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\left(\frac{\partial U}{\partial z} + \frac{\partial \overline{U}}{\partial \bar{z}}\right), \\ \frac{\partial \theta}{\partial x} - i\frac{\partial \theta}{\partial y} &= 2\frac{\partial \theta}{\partial z} = 2\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z} + \frac{\partial \overline{U}}{\partial \bar{z}}\right). \end{aligned}\right\} \tag{9.109}$$

Introducing (9.108) and (9.109) in equations (9.105), we represent them in the following form:

$$\frac{\partial}{\partial \bar{z}}\left[(k + 1)\left(\frac{\partial U}{\partial z} + \frac{\partial \overline{U}}{\partial \bar{z}}\right) + 2\frac{\partial U}{\partial z}\right] = 0,$$

$$\frac{\partial}{\partial z}\left[(k + 1)\left(\frac{\partial U}{\partial z} + \frac{\partial \overline{U}}{\partial \bar{z}}\right) + 2\frac{\partial \overline{U}}{\partial \bar{z}}\right] = 0.$$

By performing integration, we obtain (after some reduction within the brackets)

$$(k+3)\frac{\partial U}{\partial z}+(k+1)\frac{\partial \bar{U}}{\partial \bar{z}}=f_1'(z);$$

$$(k+1)\frac{\partial U}{\partial z}+(k+3)\frac{\partial \bar{U}}{\partial \bar{z}}=f_2'(\bar{z}),$$

where $f_1'(z)$ and $f_2'(\bar{z})$ are arbitrary analytic functions of the variables indicated in the parentheses. Primes denote their derivatives. We solve the last equations with respect to $\partial U/\partial z$ and $\partial \bar{U}/d\bar{z}$:

$$\left.\begin{aligned}\frac{\partial U}{\partial z}&=\frac{1}{4k+8}\left[(k+3)f_1'(z)-(k+1)f_2'(\bar{z})\right],\\ \frac{\partial \bar{U}}{\partial \bar{z}}&=\frac{1}{4k+8}\left[(k+3)f_2'(\bar{z})-(k+1)f_1'(z)\right].\end{aligned}\right\}\tag{9.110}$$

By integrating these equations, we obtain

$$\left.\begin{aligned}U&=\frac{1}{4k+8}\left[(k+3)f_1(z)-(k+1)zf_2'(\bar{z})+\varphi_2(\bar{z})\right],\\ \bar{U}&=\frac{1}{4k+8}\left[(k+3)f_2(\bar{z})-(k+1)\bar{z}f_1'(z)+\varphi_1(z)\right],\end{aligned}\right\}\tag{9.111}$$

where $\varphi_1(z)$ and $\varphi_2(\bar{z})$ are two new arbitrary analytic functions of the variables z and $\bar{z}$ (respectively). Expressions (9.111) give the general solution of Lamé's equations (9.103); if U and $\bar{U}$ are known, we can find the displacement components u and v according to (9.106):

$$u=\frac{U+\bar{U}}{2},$$

$$v=\frac{U-\bar{U}}{2i}=i\,\frac{\bar{U}-U}{2}.$$

In this way we obtain

$$\left.\begin{aligned}u=\frac{1}{8(k+2)}&\{(k+3)\left[f_1(z)+f_2(\bar{z})\right]-\\ &-(k+1)\left[zf_2'(\bar{z})+\bar{z}f_1'(z)\right]+\varphi_1(z)+\varphi_2(\bar{z})\},\\ v=\frac{i}{8(k+2)}&\{(k+3)\left[f_2(\bar{z})-f_1(z)\right]-\\ &-(k+1)\left[\bar{z}f_1'(z)-zf_2'(\bar{z})\right]+\varphi_1(z)-\varphi_2(\bar{z})\}.\end{aligned}\right\}\tag{9.112}$$

In addition to these formulas let us write the expression of the volume expansion by using (9.107) and (9.110):

$$\theta=\frac{1}{2(k+2)}\left[f_1'(z)+f_2'(\bar{z})\right].\tag{9.113}$$

To determine the components of stress we take into consideration Hooke's law in the form of equations (9.52):

$$\left.\begin{aligned} X_x &= \mu\left(2\frac{\partial u}{\partial x}+k\theta\right), \\ Y_y &= \mu\left(2\frac{\partial v}{\partial y}+k\theta\right), \\ Z_z &= \mu k\theta, \\ X_y &= \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right). \end{aligned}\right\} \tag{9.114}$$

By introducing here the values of u, v and θ from (9.112) and (9.113) we obtain

$$\left.\begin{aligned} X_x &= \frac{\mu}{4(k+2)}\{2(k+1)[f_1'(z)+f_2'(\bar{z})]- \\ &\qquad -(k+1)[zf_2''(\bar{z})+\bar{z}f_1''(z)]+\varphi_1'(z)+\varphi_2'(\bar{z})\}, \\ Y_y &= \frac{\mu}{4(k+2)}\{2(k+1)[f_1'(z)+f_2'(\bar{z})]+ \\ &\qquad +(k+1)[zf_2''(\bar{z})+\bar{z}f_1''(z)]-\varphi_1'(z)-\varphi_2'(\bar{z})\}, \\ X_y &= \frac{\mu i}{4(k+1)}\{(k+1)[-\bar{z}f_1''(z)+zf_2''(\bar{z})]+ \\ &\qquad +\varphi_1'(z)-\varphi_2'(\bar{z})\}. \end{aligned}\right\} \tag{9.115}$$

The right-hand members of formulas (9.112), (9.113) and (9.115) are, in general, functions of the complex variables z and $\bar{z}$, but according to the meaning of the problem they must be real. This will be achieved if we adjust the four so far arbitrary analytic functions

$$f_1(z), \quad f_2(\bar{z}), \quad \varphi_1(z) \text{ and } \varphi_2(\bar{z}) \tag{9.116}$$

so as to have

$$\left.\begin{aligned} f_2(\bar{z}) &= \bar{f}_1(\bar{z}), \\ \varphi_2(\bar{z}) &= \bar{\varphi}_1(\bar{z}), \end{aligned}\right\} \tag{9.117}$$

where $\bar{f}_1(\bar{z})$ and $\bar{\varphi}_1(\bar{z})$ are the conjugate expressions to $f_1(z)$ and $\varphi_1(z)$ respectively, i.e., if

$$\left.\begin{aligned} f_1(z) &= \chi(x, y)+i\psi(x, y), \\ \varphi_1(z) &= \chi_1(x, y)+i\psi_1(x, y), \end{aligned}\right\} \tag{9.118'}$$

then

$$\left.\begin{aligned} f_2(\bar{z}) &= \bar{f}_1(\bar{z}) = \chi(x, y)-i\psi(x, y), \\ \varphi_2(\bar{z}) &= \bar{\varphi}_1(\bar{z}) = \chi_1(x, y)-i\psi_1(x, y). \end{aligned}\right\} \tag{9.118''}$$

Indeed, it can easily be shown that if conditions (9.117) are satisfied, the displacements u and v defined by equalities (9.112) and, consequently, stresses (9.115) will be real quantities.

In fact,

$$\left.\begin{aligned} f_1'(z) &= \frac{\partial\chi}{\partial x} + i\frac{\partial\psi}{\partial x}, \\ f_2'(z) &= \frac{\partial\chi}{\partial x} - i\frac{\partial\psi}{\partial x}, \end{aligned}\right\} \qquad (9.119)$$

and, using (9.97), we readily find that

$$zf_2'(\bar{z}) + zf_1'(z) = 2\left(x\frac{\partial\chi}{\partial x} + y\frac{\partial\psi}{\partial x}\right);$$

$$\bar{z}f_1'(z) - zf_2'(\bar{z}) = 2i\left(x\frac{\partial\psi}{\partial x} - y\frac{\partial\chi}{\partial x}\right).$$

Substituting all this in (9.112) and (9.113), we obtain after cancellation and simplification

$$\left.\begin{aligned} u &= \frac{1}{4(k+2)}\left[(k+3)\chi(x,\ y) - (k+1)\left(x\frac{\partial\chi}{\partial x} + y\frac{\partial\psi}{\partial x}\right) + \chi_1\right], \\ v &= \frac{1}{4(k+2)}\left[(k+3)\psi(x,\ y) + (k+1)\left(x\frac{\partial\psi}{\partial x} - y\frac{\partial\chi}{\partial x}\right) - \psi_1\right], \\ \theta &= \frac{1}{k+2}\frac{\partial\chi}{\partial x}, \end{aligned}\right\} \qquad (9.120)$$

where functions χ, ψ, χ_1, ψ_1 are connected by the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial\chi}{\partial x} &= \frac{\partial\psi}{\partial y}; & \frac{\partial\chi_1}{\partial x} &= \frac{\partial\psi_1}{\partial y}; \\ \frac{\partial\chi}{\partial y} &= -\frac{\partial\psi}{\partial x}; & \frac{\partial\chi_1}{\partial y} &= -\frac{\partial\psi_1}{\partial x}. \end{aligned} \qquad (9.121)$$

Thus formulas (9.120), subject to condition (9.117), give real values for u, v and θ, as we set out to prove. It is evident that formulas (9.115) will give real values for the components of stress. Because of conditions (9.121) all the functions χ, ψ, χ_1, ψ_1, involved here are harmonic.

Formulas (9.120), subject to conditions (9.121), give the general solution of the plane problem for the case of plane strain in terms of functions of real variables x, y.

One can, however, proceed with the solution in terms of functions of the complex variables z and $\bar{z}$ taking as a basis the foregoing formulas (9.111)-(9.115) and (9.117). Indeed, one can thus easily obtain the fundamental formulas of the methods of A. E. H. Love and N. I. Muskhelishvili mentioned at the end of Sec. 64.

In addition to the last of formulas (9.120) we derive the expression of the component of rotation

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

By using formulas (9.112), (9.117) and (9.118), we get

$$2\omega_z = \frac{i}{2}\left[\bar{f}_1'(\bar{z}) - f_1'(z)\right] = \frac{i}{2}\left(-2i\frac{\partial \psi}{\partial x}\right) = \frac{\partial \psi}{\partial x}.$$

Thus,

$$(k+2)\,\theta + i2\omega_z = \frac{\partial \chi}{\partial x} + i\frac{\partial \psi}{\partial x},$$

or, on the basis of (9.119), we find

$$(k+2)\,\theta + 2i\omega_z = f_1'(z). \tag{9.122}$$

This formula was obtained by A.E.H. Love from Lamé's equations (9.103) and is fundamental in his method.

In Muskhelishvili's method the starting points are:

(1) the first of formulas (9.111) which, on the basis of (9.117), takes the form

$$u + iv = \frac{1}{4(k+2)}\left[(k+3) f_1(z) - (k+1)\, z\bar{f}_1'(\bar{z}) + \bar{\varphi}_1(\bar{z})\right]; \tag{9.123}$$

(2) the following two formulas which can easily be obtained from (9.115):

$$\left.\begin{aligned} X_x + Y_y &= \mu\frac{k+1}{k+2}\left[f_1'(z) + \bar{f}_1'(\bar{z})\right], \\ Y_y - X_x + 2iX_y &= \frac{\mu}{k+2}\left[(k+1)\,\bar{z} f_1''(z) - \varphi_1'(z)\right]. \end{aligned}\right\} \tag{9.124}$$

The first of these formulas allows us to find the complex expression of Airy's stress function employed in Chapter VI. Indeed, the left-hand member of this formula represents the Laplacian operation upon the stress function:

$$X_x + Y_y = \nabla^2\varphi.$$

Its complex expression is given in formula (9.101); on this basis we have

$$\frac{\partial^2\varphi}{\partial z\,\partial\bar{z}} = \mu\,\frac{k+1}{4(k+2)}\left[f_1'(z) + \bar{f}_1'(\bar{z})\right].$$

Double integration yields

$$\varphi = \mu\frac{k+1}{4(k+2)}\left[\bar{z} f_1(z) + z\bar{f}_1(\bar{z}) + \omega_1(\bar{z}) + \omega_2(z)\right],$$

where $\omega_1(\bar{z})$ and $\omega_2(z)$ are arbitrary functions; it is easy to verify that the bracketed expression is a real quantity; in order that the

sum of the last two terms be real we must have

$$\omega_1(\bar{z}) = \bar{\omega}_2(\bar{z}).$$

Thus the expression of the stress function is

$$\varphi = \mu \frac{k+1}{4(k+2)} [\bar{z} f_1(z) + z\bar{f}_1(\bar{z}) + \omega_2(z) + \bar{\omega}_2(\bar{z})]. \quad (9.124')$$

Let us clarify the meaning of the functions $\omega_2(z)$ and $\bar{\omega}_2(\bar{z})$ involved; to this end we first find the stress Y_y from this formula:

$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = \frac{\mu(k+1)}{4(k+2)} \{2[f_1'(z) + \bar{f}_1'(\bar{z})] + \bar{z} f_1''(z) + z\bar{f}_1''(\bar{z}) + \omega_2''(z) + \bar{\omega}_2''(\bar{z})\},$$

and then we obtain the same quantity according to the second of formulas (9.115), taking into account (9.118') and 9.118''):

$$Y_y = \frac{\mu(k+1)}{4(k+2)} \left\{ 2[f_1'(z) + \bar{f}_1'(\bar{z})] + \bar{z} f_1''(z) + z\bar{f}_1''(\bar{z}) - \frac{\varphi_1'(z)}{k+1} - \frac{\bar{\varphi}_1'(\bar{z})}{k+1} \right\}.$$

Comparing the expressions obtained, we find that

$$\omega_2''(z) = -\frac{\varphi_1'(z)}{k+1}; \quad \omega_2'(z) = -\frac{\varphi_1(z)}{k+1}.$$

Formulas (9.123), (9.124) and (9.124') will coincide in this condition with those given by N. I. Muskhelishvili if we replace the functions f_1 and φ_1 introduced here by other functions φ and ψ according to the formulas

$$\left.\begin{aligned} \frac{k+1}{2(k+2)} \mu f_1(z) &= \varphi(z), \\ -\frac{1}{2(k+2)} \mu\varphi_1(z) &= \psi(z) \end{aligned}\right\} \quad (9.125)$$

and if we set

$$\frac{k+3}{k+1} = 3 - 4\sigma = \varkappa.$$

All our reasoning pertains to the case of plane strain; for the case of generalised plane stress Hooke's law takes the form

$$\begin{aligned} X_x &= \mu(2e_{xx} + k\theta), \\ Y_y &= \mu(2e_{yy} + k\theta), \\ Z_z &= 0 = \mu(2e_{zz} + k\theta). \end{aligned}$$

Hence

$$e_{zz} = -\frac{k}{2}\theta; \quad \theta = e_{xx} + e_{yy} - \frac{k}{2}\theta;$$

$$\theta = \frac{2}{k+2}(e_{xx} + e_{yy}) = \frac{2}{k+2}\theta',$$

where

$$\theta' = e_{xx} + e_{yy}.$$

Therefore we have

$$\left.\begin{aligned} X_x &= \mu(2e_{xx} + k'\theta'), \\ Y_y &= \mu(2e_{yy} + k'\theta'), \end{aligned}\right\} \tag{9.126}$$

where

$$k' = \frac{2k}{k+2} = \frac{2\sigma}{1-\sigma}.$$

Thus in the case of generalised plane stress all formulas hold good if the elastic constant k is replaced in them by k'.

66 WAVE EQUATIONS

In Sections 26-28 of Chapter IV we established that Lamé's dynamic equations

$$\left.\begin{aligned} (\lambda+\mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u &= \rho\frac{\partial^2 u}{\partial t^2}, \\ (\lambda+\mu)\frac{\partial\theta}{\partial y} + \mu\nabla^2 v &= \rho\frac{\partial^2 v}{\partial t^2}, \\ (\lambda+\mu)\frac{\partial\theta}{\partial z} + \mu\nabla^2 w &= \rho\frac{\partial^2 w}{\partial t^2} \end{aligned}\right\} \tag{9.127}$$

permitted the solution of problems involving small motions of an elastic body and considered the simplest cases of propagation of small uniform vibrations of an infinite elastic medium, described by the differential equation of the form

$$\frac{\partial^2 f}{\partial t^2} = a^2\frac{\partial^2 f}{\partial x^2}, \tag{9.128}$$

where f is a function of x and t. This is the simplest form of a "wave equation"[1]. To conclude the present chapter we derive wave equations for the general problem of propagation of elastic vibrations assuming, as before, that body forces are absent; therefore, they are omitted in equations (9.127).

Let us apply to equations (9.127) the procedure that was used in Sec. 36 to obtain equation (5.55), i.e., we differentiate them with respect to x, y and z, respectively, and add; we take into account that (for constant density ρ)

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial^2 f}{\partial t^2}\right) = \rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial f}{\partial x}\right);$$

---

[1] Cf. equations, (4.8), (4.11), (4.26).

the other two equalities of the same form are obtained replacing x by y and by z; we get

$$(\lambda+2\mu)\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right)=\rho\frac{\partial^2\theta}{\partial t^2}, \tag{9.129}$$

or

$$\frac{\partial^2\theta}{\partial t^2}=a^2\nabla^2\theta, \tag{9.129'}$$

where, as in (4.9), we designate

$$a^2=\frac{\lambda+2\mu}{\rho}. \tag{9.130}$$

We now differentiate the third equation (9.127) with respect to y and the second with respect to z and subtract one result from the other:

$$\mu\nabla^2\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right);$$

but, according to (2.9),

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=2\omega_x,$$

and, therefore, the preceding equation reduces to the previous form (9.129)

$$\frac{\partial^2\omega_x}{\partial t^2}=b^2\nabla^2\omega_x, \tag{9.131a}$$

where [cf. formula (4.12)]

$$b^2=\frac{\mu}{\rho}. \tag{9.132}$$

A cyclic change in (9.131a) gives two equations of the same form:

$$\left.\begin{aligned}\frac{\partial^2\omega_y}{\partial t^2}&=b^2\nabla^2\omega_y,\\ \frac{\partial^2\omega_z}{\partial t^2}&=b^2\nabla^2\omega_z.\end{aligned}\right\} \tag{9.131b}$$

Equations having the structure of (9.129) and (9.131a, b) represent "wave equations" of the general form. The coefficients a and b, as we saw in Sec. 26, represent the velocities of propagation of plane waves of two kinds: waves of dilatation and waves of distortion. If we succeed in integrating equations (9.129) and (9.131a, b), there will be determined everywhere: the volume expansion θ, i.e., the divergence of the displacement vector $\mathbf{u}(u, v, w)$, and the solenoidal vector (curl) $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$.

In vector analysis it is proved that, if θ and ω are known, one can find the vector $\mathbf{u}$ at every point of the field, i.e., one can determine the displacements of the points of an elastic space [1].

Consider two limiting cases separately:

1. $\theta=0$ throughout the space;
2. $\omega_x=\omega_y=\omega_z=0$ throughout the space.

In the first case the wave process takes place without change of volume; this, as is seen directly from Lamé's equations (9.127), results in wave equations of the general form for the components of the displacement vector:

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= b^2\nabla^2 u, \\ \frac{\partial^2 v}{\partial t^2} &= b^2\nabla^2 v, \\ \frac{\partial^2 w}{\partial t^2} &= b^2\nabla^2 w, \end{aligned}\right\} \tag{9.133}$$

where notation (9.132) is introduced. Waves of this type are called equivoluminal (variously called distortional, rotational, shear, transverse, and, in seismology, *S* waves); they travel with the velocity

$$b=\pm\sqrt{\frac{\mu}{\rho}}.$$

The second case $\omega_x=\omega_y=\omega_z=0$ occurs when the displacements u, v, w have a potential (see the end of Sec. 10 of Chapter II)

$$u=\frac{\partial\Phi}{\partial x};\quad v=\frac{\partial\Phi}{\partial y};\quad w=\frac{\partial\Phi}{\partial z}.$$

Proceeding from these relations, we transform the first of Lamé's equations (9.127):

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=\nabla^2\Phi;$$

therefore,

$$\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial x}\nabla^2\Phi=\nabla^2\left(\frac{\partial\Phi}{\partial x}\right)=\nabla^2 u.$$

It remains to substitute this in the first of equations (9.127); making an analogous transformation in the other two equations

[1] See N. E. Kochin, *Vector Analysis*, Moscow, Objed. nauchno-tekhn. izd., 1938, p. 223.

and introducing notation (9.130), we obtain

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= a^2\nabla^2 u,\\ \frac{\partial^2 v}{\partial t^2} &= a^2\nabla^2 v,\\ \frac{\partial^2 w}{\partial t^2} &= a^2\nabla^2 w.\end{aligned}\right\} \tag{9.134}$$

We have again wave equations of the general form for the displacement, just as in the case of equivoluminal waves (9.133); waves, corresponding to the case under consideration, are called irrotational waves of dilatation (variously called dilatational, bulk, longitudinal, and, in seismology, P waves). Their velocity of propagation a, as we already know from Sec. 26, is considerably greater than that of equivoluminal waves b. It is clear from the foregoing that the fundamental problem of the theory of elastic waves is the problem of integrating the general wave equation

$$\frac{\partial^2 f(x, y, z, t)}{\partial t^2} = c^2\nabla^2 f, \tag{9.135}$$

where

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}; \tag{9.136}$$

c is the velocity of wave propagation. The simplest wave equations of uniform vibrations (plane waves), treated in Sections 26 and 28, result from (9.135) if f depends on one co-ordinate only, for example, on x, since then, according to (9.136),

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2},$$

and in place of (9.135) we get

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}. \tag{9.137}$$

For this equation we have D'Alembert's general solution [see formula (4.18)]

$$f = \varphi(x - ct) + \psi(x + ct), \tag{9.138}$$

where φ and ψ are symbols of arbitrary functions.

67. SOME PARTICULAR SOLUTIONS OF THE WAVE EQUATION

We cannot dwell here on the general problem of integrating the wave equation (9.135) but we may point out one simple property of its particular solutions which is quite analogous to the property of particular solutions of Laplace's equations indicated at the

beginning of Sec. 58; if any particular solution of equation (9.135) is known

$$f = f_0(x, y, z, t, \alpha, \beta, \gamma, \ldots), \tag{9.139}$$

where α, β, γ, ... are some parameters or coefficients, then the partial derivative of f_0 with respect to any of the arguments indicated in the parentheses of (9.139) will also be a particular solution of equation (9.135). The validity of this conclusion follows directly from the fact that if ζ denotes any of the above arguments, then

$$\left.\begin{aligned} \frac{\partial}{\partial \zeta}\left(\frac{\partial^2 f_0}{\partial t^2}\right) &= \frac{\partial^2}{\partial t^2}\left(\frac{\partial f_0}{\partial \zeta}\right); \\ \frac{\partial}{\partial \zeta}(\nabla^2 f_0) &= \nabla^2\left(\frac{\partial f_0}{\partial \zeta}\right). \end{aligned}\right\} \tag{9.140}$$

Indeed, since f_0 is a solution of equation (9.135), then

$$\frac{\partial^2 f_0}{\partial t^2} = c^2 \nabla^2 f_0.$$

Differentiating both members with respect to ζ, we have

$$\frac{\partial}{\partial \zeta}\left(\frac{\partial^2 f_0}{\partial t^2}\right) = c^2 \frac{\partial}{\partial \zeta}(\nabla^2 f_0),$$

and on the basis of (9.140) we get

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial f_0}{\partial \zeta}\right) = c^2 \nabla^2 \left(\frac{\partial f_0}{\partial \zeta}\right);$$

hence

$$\frac{\partial f_0}{\partial \zeta}$$

is a solution of equation (9.135), as we set out to prove.

Consider an extremely important particular solution of the form

$$f = f_0(r, t), \tag{9.141}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ represents the radius-vector of an arbitrary point with respect to the origin. It is obvious that solution (9.141) corresponds to a symmetrical distribution of factor f_0 with respect to the origin, so that the quantity f_0 [1] has the same value at all points of a sphere of radius r with centre at the origin; this is a case of spherical waves having a source at a point taken as the origin. Such waves are considered here only as a mathematical image [i.e., as a particular solution of equation (9.135)]; however, they can really be produced as a result of the action of a momentary or, in general, of a varying force applied at the origin.

[1] By f_0 is meant any of the quantities

$$u, v, w, \omega_x, \omega_y, \omega_z, \Theta.$$

Let us check the possibility of solution (9.141) by substituting it in (9.135), and to that end calculate Laplace's operator $\nabla^2 f_0$ [cf. formulas (9.25)-(9.27)]:

$$\frac{\partial f_0}{\partial x}=\frac{\partial f_0}{\partial r}\frac{\partial r}{\partial x}=\frac{\partial f_0}{\partial r}\frac{x}{r},$$

$$\frac{\partial^2 f_0}{\partial x^2}=\frac{\partial^2 f_0}{\partial r^2}\frac{x^2}{r^2}+\frac{\partial f_0}{\partial r}\frac{r-\frac{x}{r}x}{r^2}=\frac{\partial^2 f_0}{\partial r^2}\frac{x^2}{r^2}+\frac{\partial f_0}{\partial r}\frac{r^2-x^2}{r^3}.$$

Calculating similarly the second derivatives with respect to y and z, we get

$$\frac{\partial^2 f_0}{\partial x^2}=\frac{\partial^2 f_0}{\partial r^2}\frac{x^2}{r^2}+\frac{\partial f_0}{\partial r}\frac{r^2-x^2}{r^3},$$

$$\frac{\partial^2 f_0}{\partial y^2}=\frac{\partial^2 f_0}{\partial r^2}\frac{y^2}{r^2}+\frac{\partial f_0}{\partial r}\frac{r^2-y^2}{r^3},$$

$$\frac{\partial^2 f_0}{\partial z^2}=\frac{\partial^2 f_0}{\partial r^2}\frac{z^2}{r^2}+\frac{\partial f_0}{\partial r}\frac{r^2-z^2}{r^3}.$$

Adding up these equalities by members and taking into account that

$$x^2+y^2+z^2=r^2,$$

we obtain

$$\nabla^2 f_0=\frac{\partial^2 f_0}{\partial r^2}+\frac{2}{r}\frac{\partial f_0}{\partial r}=\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf_0).$$

Substituting this in (9.135), we have

$$\frac{\partial^2 f_0}{\partial t^2}=\frac{c^2}{r}\frac{\partial^2}{\partial r^2}(rf_0),$$

or

$$r\frac{\partial^2 f_0}{\partial t^2}=c^2\frac{\partial^2}{\partial r^2}(rf_0);$$

but it is evident that

$$r\frac{\partial^2 f_0}{\partial t^2}=\frac{\partial^2}{\partial t^2}(rf_0),$$

and the last equation takes the form of the simplest wave equation [cf. (9.137)]:

$$\frac{\partial^2}{\partial t^2}(rf_0)=c^2\frac{\partial^2}{\partial r^2}(rf_0);$$

therefore, we immediately write its general solution of the type of (9.138):

$$rf_0=\varphi(r-ct)+\psi(r+ct),$$

where φ and ψ are arbitrary functions. Hence we obtain the required particular solution of equation (9.135), i.e., the equation

of a spherical wave:

$$f_0 = \frac{1}{r}\varphi(r - ct) + \frac{1}{r}\psi(r + ct). \tag{9.142}$$

The first term on the right-hand side corresponds to a wave propagated from the centre (the origin), the second term to a wave travelling in the opposite direction (the reflected wave). Solution (9.142), like solution (9.138) for the case of uniform vibrations, represents a moving wave.

Very important particular solutions of the "standing wave" type may be obtained for the general equation (9.135) by the method of separation of variables, which was applied to the simplest wave equation (4.28) in Sec. 28. For that it is necessary to seek solutions in the form of a product of two functions

$$f = F(x, y, z) \cdot T(t). \tag{9.143}$$

Introducing this in (9.135), we obtain

$$F(x, y, z)\, T''(t) = c^2\nabla^2 F(x, y, z)\, T(t)$$

or, dividing both members by $F \cdot T$, we separate the variables

$$\frac{T''(t)}{T(t)} = c^2 \frac{\nabla^2 F(x, y, z)}{F(x, y, z)} = -\lambda^2,$$

where $-\lambda^2$ is an arbitrary constant. This leads to two differential equations:

$$T''(t) + \lambda^2 T(t) = 0, \tag{9.144}$$

$$\nabla^2 F(x, y, z) + \frac{\lambda^2}{c^2} F(x, y, z) = 0. \tag{9.145}$$

The particular solutions of equation (9.144)

$$T_1 = \sin \lambda t; \quad T_2 = \cos \lambda t$$

give harmonic vibrations; therefore, solution (9.143) takes one of the forms:

$$f_1 = F(x, y, z) \sin \lambda t,$$

$$f_2 = F(x, y, z) \cos \lambda t,$$

where $F(x, y, z)$ must satisfy equation (9.145).

The dilatational and equivoluminal waves discussed here take place in the interior of an elastic body; in the neighbourhood of its surface waves of a different type are possible (Rayleigh waves), while near the surface of contact of two elastic bodies special waves occur (Love waves) [1].

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[1] See, for instance, L. S. Leibenzon, *Theory of Elasticity*, 2nd ed., Moscow. Gostekhizdat, 1947.

X

Bending of a Plate

68. GENERAL

A plate is a body of prismatical or cylindrical form whose height h is small compared with the dimensions of the faces (Fig. 94).

Let us mark off the middle plane of the plate which divides its height in half. This plane plays the same role in the theory of plates as the axis and the neutral plane in the theory of bending of rods. Let us specify the system of co-ordinates, placing the

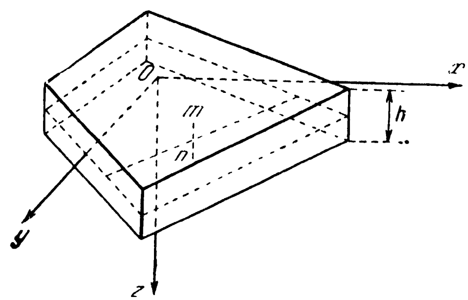

Fig. 94

axes Ox and Oy in the middle plane; the axis Oz is directed downward.

In Chapters VI and VII we discussed the problems related to stresses and deformations of such a plate in the case when the external load and the corresponding reactions act in the middle plane (generalised plane stress). If the load does not exceed a certain limit, the middle plane does not deflect. We shall now turn to another case where the load and the supporting reactions are normal to the middle plane; in this case, the middle plane of the

plate deflects, and we deal with the bending of a flat plate. It should be noted that, in general, we encounter here not only bending (as it is understood in the elementary theory of bending of a beam), since in a great majority of cases bending is accompanied by torsion of a plate.

Below we consider the approximate theory of plate bending, which is valid for plates whose thickness is small compared with their other dimensions but whose deflections are small compared with the thickness. Such plates are called "moderately thick plates" or, in Galerkin's terminology, "thin slabs". In the study of stresses and deformations of such plates one may employ the same methods as in the approximate theory of bending of rods, but in this case the theory of bending is considerably complicated both by the above-indicated phenomenon of twisting and by the fact that one has to relate the stress and strain distribution to a whole plane (the middle plane) rather than to a single line (the axis of a rod). This leads to the replacement of ordinary differential equations of bending by partial differential equations whose integration is much more involved.

Henceforth, to simplify the theory, we shall make the following assumptions:

1. In the analysis of strains we disregard (just as in the elementary theory of bending of rods) the stresses Z_z, which occur on account of the mutual pressure of horizontal layers of the plate, and, consequently, the strains e_{zz} of the plate in the direction of its thickness; because of this we take the generalised Hooke's law (V) for a plate in the following form:

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E}(X_x - \sigma Y_y), \\ e_{yy} &= \frac{1}{E}(Y_y - \sigma X_x). \end{aligned}\right\} \qquad (10.1)$$

2. We introduce a hypothesis analogous to the hypothesis of plane sections in a rod. If a plate is bent into a cylindrical surface, the above hypothesis may be accepted in the form as it is formulated for a rod: plane cross sections of a plate subjected to bending remain plane and normal to the deflected middle plane. If, however, the plate is not bent into a cylindrical surface, our hypothesis will be formulated as: a linear element mn (Fig. 94) within a plate perpendicular to the middle plane remains straight and normal to this plane after its deflection during bending. This assumption was first proposed by G. Kirchhoff; it is sometimes called the *hypothesis of linear elements*.

3. Since the deflections of a plate are small, we assume that the points of the middle plane displace only in the direction of

the z axis, i.e., for them

$$u=0; \quad v=0; \quad w=f(x, y).$$

This obviously implies that deformations, i.e., extensions and shears, are absent in the middle plane.

69. BASIC EQUATIONS OF BENDING AND TORSION OF A PLATE

The foregoing assumptions make it possible to derive the basic equations of the approximate theory of the bending of moderately thick plates. According to the third of these assumptions, the displacements u and v vanish in the middle plane; they are different

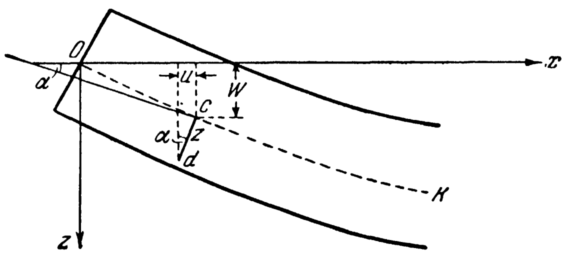

Fig. 95

from zero at points outside this plane, but Kirchhoff's hypothesis enables us to express them in terms of deflections $w(x, y)$ of the points of the middle plane. Indeed, Fig. 95 shows a section of the plate by a plane parallel to Oxz. Let OK be the trace of the middle plane after deflection. Let us take any point d at a distance $cd=z$ from the middle plane. Taking into account the second and third points of the assumptions made above, we have from the drawing

$$u=-z\sin\alpha\approx -z\tan\alpha=-z\frac{\partial w}{\partial x},$$

where α is the angle of inclination of the tangent to the trace of the middle plane; we take the minus sign in the equality because, for instance, in the case represented in Fig. 95 $z>0$, $\tan\alpha>0$, $u<0$. Thus

$$u=-z\frac{\partial w}{\partial x}. \tag{10.2}$$

Making a section of the plate by a plane parallel to Oyz, we find by the same reasoning

$$v=-z\frac{\partial w}{\partial y}. \tag{10.3}$$

With relations (10.2) and (10.3) we can obtain the expressions of the following three components of the strain tensor from Cauchy's formulas (III):

$$\left.\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2};\\ e_{yy} &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2};\\ e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2z\frac{\partial^2 w}{\partial x\,\partial y}. \end{aligned}\right\} \tag{10.4}$$

We now solve the equations of Hooke's law (10.1) for stresses

$$X_x = \frac{E}{1-\sigma^2}(e_{xx} + \sigma e_{yy}),$$

$$Y_y = \frac{E}{1-\sigma^2}(e_{yy} + \sigma e_{xx}),$$

$$Y_x = X_y = \frac{E}{2(1+\sigma)}e_{xy} = \frac{E}{2(1-\sigma^2)}(1-\sigma)\,e_{xy}.$$

Substituting here the values of e_{xx}, e_{yy}, e_{xy} from (10.4), we obtain the expressions of three of the components of the stress tensor in terms of deflection $w(x, y)$:

$$\left.\begin{aligned} X_x &= -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right),\\ Y_y &= -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right),\\ Y_x = X_y &= -\frac{Ez}{1-\sigma^2}(1-\sigma)\frac{\partial^2 w}{\partial x\,\partial y}. \end{aligned}\right\} \tag{10.5}$$

The remaining three components X_z, Y_z, Z_z can now be obtained from the equations of equilibrium (I) which have not yet been utilised; the first of them gives

$$\frac{\partial X_z}{\partial x} = -\frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} = \frac{Ez}{1-\sigma^2}\left[\frac{\partial^3 w}{\partial x^3} + \sigma\frac{\partial^3 w}{\partial y^2\,\partial x} + (1-\sigma)\frac{\partial^3 w}{\partial x\,\partial y^2}\right] =$$

$$= \frac{Ez}{1-\sigma^2}\frac{\partial}{\partial x}\nabla^2 w,$$

where

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$$

In the same way we obtain from the second equation of (I)

$$\frac{\partial Y_z}{\partial z} = \frac{Ez}{1-\sigma^2}\frac{\partial}{\partial y}\nabla^2 w.$$

It remains to integrate these two equations with respect to z, taking into account that

$$\frac{\partial}{\partial x}\nabla^2 w \quad \text{and} \quad \frac{\partial}{\partial y}\nabla^2 w$$

are independent of z; we have

$$\int z\,dz = \frac{z^2}{2} + C. \tag{10.6}$$

The loads and the supporting reactions of the plate are assumed to be normal to its upper and lower planes and, therefore, on these planes, i.e., for $z = \pm\frac{h}{2}$ there should be

$$X_z = Y_z = 0.$$

We chose accordingly the value of the arbitrary constant in formula (10.6):

$$C = -\frac{h^2}{8},$$

and obtain the final expressions of the shearing stresses:

$$\left.\begin{aligned} X_z &= \frac{E\left(z^2 - \frac{h^2}{4}\right)}{2(1-\sigma^2)}\frac{\partial}{\partial x}\nabla^2 w, \\ Y_z &= \frac{E\left(z^2 - \frac{h^2}{4}\right)}{2(1-\sigma^2)}\frac{\partial}{\partial y}\nabla^2 w. \end{aligned}\right\} \tag{10.7}$$

Turning, at last, to the third equation of (I), we find

$$\frac{\partial Z_z}{\partial z} = -\frac{\partial X_z}{\partial x} - \frac{\partial Y_z}{\partial y}$$

or, substituting here the expressions of the shearing stresses from (10.7),

$$\frac{\partial Z_z}{\partial z} = -\frac{E}{2(1-\sigma^2)}\left(z^2 - \frac{h^2}{4}\right)\nabla^2\nabla^2 w. \tag{10.8}$$

Hence by integrating with respect to z, we can obtain the expression of Z_z, which characterises the mutual pressure of horizontal layers of the plate; however, the stress Z_z in itself is of no interest because in absolute value it does not exceed the intensity of the external load on the plate; the latter, however, is negligibly small in comparison with the stresses (10.5) set up by the bending of the plate and, therefore, Z_z was disregarded in the formulas of Hooke's law (10.1). Equation (10.8) will be used for another, more important purpose; by integrating it throughout the thickness of

the plate, i.e., from $-\frac{h}{2}$ to $+\frac{h}{2}$, we eliminate z from it and thus obtain a differential equation for determining the function $w(x, y)$ through which it was found possible to express the remaining displacements [formulas (10.2) and (10.3)] and the stresses [formulas (10.5) and (10.7)].

Taking into account that

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(z^2 - \frac{h^2}{4}\right) dz = -\frac{h^3}{6}, \tag{10.8'}$$

we obtain from (10.8)

$$Z_z\left(\frac{h}{2}\right) - Z_z\left(-\frac{h}{2}\right) = \frac{Eh^3}{12(1-\sigma^2)} \nabla^2\nabla^2 w. \tag{10.9}$$

It is assumed that the load $q(x, y)$ is applied on the upper surface of the plate $\left(z = -\frac{h}{2}\right)$; then for the chosen system of coordinates (Fig. 94) we have

$$q(x, y) = Z_{-z}\left(-\frac{h}{2}\right) = -Z_z\left(-\frac{h}{2}\right),$$

$$Z_z\left(\frac{h}{2}\right) = 0.$$

Substituting these values in (10.9), we obtain the required differential equation with respect to deflection $w(x, y)$:

$$\nabla^2\nabla^2 w = \frac{q(x, y)}{D}. \tag{10.10}$$

Here [see formula (6.5)]

$$D = \frac{Eh^3}{12(1-\sigma^2)} = \frac{E_1 h^3}{12}. \tag{10.11}$$

This quantity is called the *flexural rigidity of a plate*, by analogy with the rigidity of a rod or a bar of rectangular cross section with dimensions b and h in the elementary theory of bending:

$$B = EJ = \frac{Ebh^3}{12} \quad \text{or} \quad \text{for} \quad b = 1 \quad B = \frac{Eh^3}{12}.$$

In the expanded form equation (10.10) is written down as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x, y)}{D}. \tag{10.10'}$$

It is sometimes called the *equation of Sophie Germain* after the young French investigator who first obtained it in the final form in 1815.

In solving each particular problem, one has to add to equation (10.10) the boundary conditions, i.e., the fixing conditions of a plate; we shall discuss them later. Now we shall proceed to clarify the physical meaning of the formulas obtained.

70. ANALYSIS OF THE RESULTS OBTAINED

Turning to formulas (10.5), we note that the second derivatives

$$\frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2}$$

give approximately the curvatures of the curves in which the middle surface of a plate is cut by planes $y=\text{const}$ and $x=\text{const}$ passed through a given point. These curvatures characterise the phenomenon of bending of a plate and show that the stresses X_x and Y_y are set up by bending; they vary in proportion to z, i.e., according to a linear law, across the thickness of the plate. The same law governs the variation of normal stresses at the cross section of a rod in the elementary theory of bending and these stresses determine the *bending moment* at the cross section of the rod. When a plate is subjected to bending there occur two bending moments at each point which act at the sections normal to the x and y axes; to calculate them we isolate an element with dimensions dx, dy, h from a plate (Fig. 96) by four vertical planes; the elementary strips of width dz on the faces of the element are acted upon by the forces

$$X_x\, dy\, dz; \quad Y_y\, dx\, dz.$$

The bending moments acting on the faces of the element will be calculated as the moments of these forces about the y and x axes:

$$M_1\, dy = \int_{-\frac{h}{2}}^{+\frac{h}{2}} X_x z\, dy\, dz = dy \int_{-\frac{h}{2}}^{+\frac{h}{2}} X_x z\, dz; \quad M_2\, dx = dx \int_{-\frac{h}{2}}^{+\frac{h}{2}} Y_y d\, zz$$

Dividing these equalities by dy and dx respectively, we obtain the bending moments in the plate per unit length of the cross section:

$$M_1 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} X_x z\, dz; \quad M_2 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} Y_y z\, dz. \tag{10.11}$$

If we substitute here the expressions of stresses (10.5) and carry out integration, then, taking notation (10.11) into account, we finally get

$$\left.\begin{aligned} M_1 &= -D\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right), \\ M_2 &= -D\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right). \end{aligned}\right\} \tag{10.12}$$

In addition to the bending moments, the faces of the isolated element are acted upon by shearing forces due to the presence of

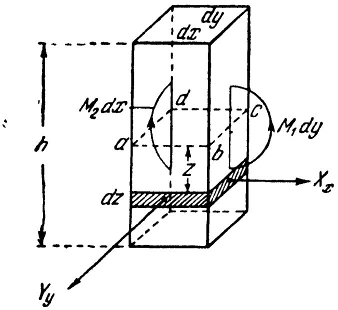

Fig. 96

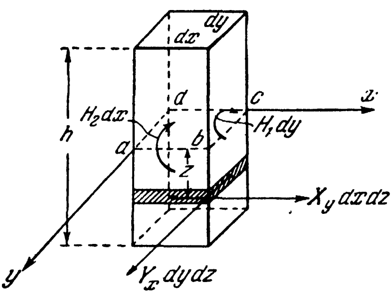

Fig. 97

the shearing stresses $Z_x = X_z$ and $Z_y = Y_z$, which are determined by formulas (10.7); induced on the faces of the element are the forces

$$Z_x\,dy\,dz; \quad Z_y\,dx\,dz.$$

The required shearing forces will be calculated as the resultants of these forces:

$$N_1\,dy = \int_{-\frac{h}{2}}^{+\frac{h}{2}} Z_x\,dy\,dz; \quad N_2\,dx = \int_{-\frac{h}{2}}^{+\frac{h}{2}} Z_y\,dx\,dz.$$

It is necessary to substitute here the expressions of stresses (10.7) and carry out integration; taking into account (10.8′) and (10.11), we obtain the expressions of the shearing forces per unit length of the section

$$N_1 = -D\frac{\partial}{\partial x}\nabla^2 w; \quad N_2 = -D\frac{\partial}{\partial y}\nabla^2 w. \tag{10.13}$$

The bending moments (10.12) in conjunction with the shearing forces (10.13) determine the phenomenon of bending of a plate.

It remains to consider the effect of the shearing stresses X_y [the third of formulas (10.5)] on the deformation of the plate.

We have seen above that the second derivatievs $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ give the curvatures of the sections of the middle surface of the plate; the mixed derivative $\frac{\partial^2 w}{\partial x\, \partial y}$ can be written down in one of the following forms:

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}(\tan \alpha_x),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}(\tan \alpha_y), \qquad \text{(a)}$$

where $\tan \alpha_x$ and $\tan \alpha_y$ are the angular coefficients of the tangents to the aforementioned sections; thus the mixed derivative is the rate of change of the slope of the tangent to the section when the latter moves in the direction normal to its plane; in this sense the mixed derivative characterises the "torsion" of the middle surface of the plate. The shearing stresses Y_x, according to (10.5), vary directly as z, i.e., linearly across the thickness of the plate; the resultant of the system of elementary forces $X_y dx\, dz$ (Fig. 97) vanishes and they will reduce to a couple with the moment[1]

$$H_1\, dy = \int_{-\frac{h}{2}}^{+\frac{h}{2}} Y_x z\, dy\, dz,$$

which is called the twisting moment. Substituting here the expression of the shearing stress Y_x from (10.5) and carrying out integration, we obtain the twisting moment per unit length of the section:

$$H_1 = -D(1-\sigma)\frac{\partial^2 w}{\partial x\, \partial y}. \qquad (10.14)$$

The shearing stresses X_y acting on the face normal to the y axis will also give the twisting moment, H_2, but it is equal to $-H_1$; this follows both from the law of reciprocity of shearing stresses $Y_x = X_y$ and from the fact that formula (10.14) is not altered by interchanging x and y.

[1] The moment is considered positive if, when viewing from the side of the positive x axis, we see the rotation in the clockwise direction; the mixed derivative, which represents the "torsion" of a surface, is positive when the tangent rotates counterclockwise [see above — the second form of formula (a)]. This is taken into account in the sign of the right-hand member of formula (10.14).

Summarising the arguments set forth in this section, we come to the conclusion that the deformation of the plate under the action of the load $q(x, y)$ applied to its upper plane is determined by the differential equation (10.10′). This deformation results from: (1) bending produced by the bending moments M_1 and M_2 [formulas (10.12)] and the shearing forces N_1 and N_2 [formulas (10.13)]; (2) torsion produced by the twisting moment $H=H_1$ [formula (10.14)]. Both of these phenomena are generally inseparable in a plate. Indeed, let us revert to the considerations of Sec. 38 in regard to the case represented in Fig. 47. If the plate is replaced by a flooring composed of separate rods, each of them will bend under the action of the load acting on it irrespective of the neighbouring rods; let them now be tied together in a solid slab (plate). If we load only one rod, whose cross section *mnpq* is shown in Fig. 47, then, deflecting, it will carry along the adjacent rods applying to their faces those shearing forces which we have designated here by N_1 and N_2; these forces will cause rotation of the cross section, i.e., the twisting of the rod.

$$M_1 = M_2 = N_1 = N_2 = 0; \quad H = \text{const.} \tag{10.15}$$

Then formula (10.14) is reduced to a differential equation in the function $w(x, y)$

$$\frac{\partial^2 w}{\partial x\, \partial y} = -\frac{H}{D(1-\sigma)}. \tag{10.16}$$

It is easy to verify by substitution that its particular solution satisfying conditions (10.15) can be taken in the form

$$w(x, y) = -\frac{H}{D(1-\sigma)}\, xy. \tag{10.17}$$

This equation indicates that the middle surface of the plate under pure torsion is a hyperbolic paraboloid (or the so-called oblique plane). It has been ascertained above that the mixed derivative of the deflection w represents the torsion of a surface which is equal to the angle of twist per unit length designated in Chapter VII by τ. If we also take into account that

$$D(1-\sigma) = \frac{Eh^3}{12(1-\sigma^2)}(1-\sigma) = \frac{\mu h^3}{6} \quad \text{[cf. formula (3.12)]},$$

then formula (10.17) will be written down as

$$\tau = -\frac{H}{\frac{\mu h^3}{6}}.$$

We consider the case where the plate is twisted by two couples $H=-H'$ per unit length of the section; if this length is denoted by c, then

$$H=\frac{M_t}{c},$$

where M_t is the twisting moment applied to the whole section, and we obtain

$$\tau=-\frac{M_t}{\frac{\mu c h^3}{6}}. \tag{10.18}$$

Suppose that a rectangular plate is twisted by torques applied to its four edges. Formula (10.18) gives the unit angle of twist. If we eliminate, for instance, the couple $-H'$, we shall have the case of simple torsion by torques considered in Chapter VIII. It should be expected that in this case the angle of twist will be one half as large as that given by (10.18), and we obtain

$$\tau=-\frac{M_t}{\frac{\mu c h^3}{3}}.$$

This result is coincident with Saint-Venant's formula (8.59), if we set $\alpha=\frac{1}{3}$ in it for the case of a very narrow rectangular section (see Table at the end of Sec. 53).

The case of *pure bending* of a plate occurs if we set

$$N_1=N_2=H=0; \quad M_1=M_2=M=\text{const}. \tag{10.19}$$

In this case formulas (10.12) give two differential equations:

$$\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}=-\frac{M}{D},$$

$$\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}=-\frac{M}{D};$$

by solving them for the derivatives, we obtain

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial y^2}=-\frac{M}{D(1+\sigma)}. \tag{10.20}$$

Their solution, satisfying conditions (10.19), is

$$w=\frac{M}{2D(1+\sigma)}(x^2+y^2). \tag{10.21}$$

It follows from this that the middle plane of the plate under pure bending is converted into a paraboloid of revolution. If the approximate values of the curvatures were replaced by the exact ones in

equations (10.20),

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{1}{\rho} = \text{const},$$

we would find directly from these equations that the middle plane is bent into a spherical surface of radius

$$\rho = \frac{D(1+\sigma)}{M}.$$

The pure bending of a plate occurs only in exceptional cases, for instance, in a circular plate loaded along its contour by uniformly distributed bending moments of intensity M per unit length of the contour.

71. BOUNDARY CONDITIONS FOR A PLATE

As pointed out earlier, the boundary conditions are the conditions on the surface of a plate which must be prescribed in advance in order to obtain the solution of equation (10.10′) corresponding to the specific problem under consideration. Such conditions include the load $q(x, y)$ on the upper and lower planes of the plate but it is taken into account in the formulation of the general problem of bending of plates and it enters in the free term of equation (10.10′). It remains to clarify the conditions on the lateral surface, i.e., at the edges of the plate, depending on the fastening or supporting conditions. For simplicity's sake, let us begin with the case of a rectangular plate (Fig. 98) whose edges are parallel to the axes Ox and Oy. Ignoring the fixing conditions indicated in the drawing, we note that at the left-hand or the right-hand edge ($x=0$ or $x=a$), for instance, there act at each point

$$\left.\begin{array}{l}\text{bending moment } M_1,\\ \text{twisting moment } H,\\ \text{shearing force } N_1,\end{array}\right\} \qquad (10.22)$$

which we can specify in advance and, moreover, we can impose the condition on the deflection w requiring that it be equal to zero or to a given value. The question of boundary conditions in the problem of bending of a plate proved to be exceedingly difficult and it was treated in detail by a number of the greatest scientists of the 19th century (S. D. Poisson, G. Kirchhoff, Kelvin and Tait, H. Lamb and others). We can elucidate it here only briefly, proceeding from the following considerations.

Let the load $q(x, y)$ be absent; in this case, the bending of the plate can nevertheless take place under the action of the forces

applied at the edges (for instance, the forces M, H, N indicated above) or as a consequence of the deflections w prescribed here. In this case equation (10.10′) becomes homogeneous

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0,$$

and coincides with equation (IX) of Chapter VI for the stress function in the plane problem of the theory of elasticity. By using it in the solution of the problem for a region bounded by a given contour, we obtain a unique, quite definite solution by specifying two conditions on this contour: *in the first problem*, the values of the normal and shearing stresses are assigned on the contour; *in*

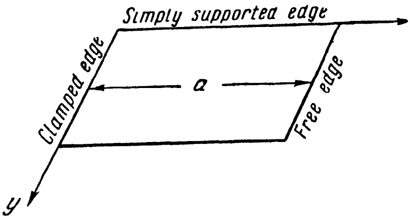

Fig. 98

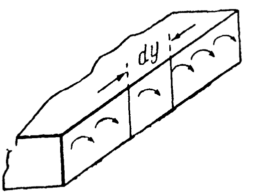

Fig. 99

the second, the displacements u and v of the points of the contour are prescribed; *in the mixed problem*, a combination of one of the displacements and one of the stresses can be prescribed (if, for instance, a strip rests on rigid foundation, the displacement normal to their surface of contact will be zero, and so will the shearing stress in the absence of friction). The kindred character of equations (10.10′) and (IX) indicates that in the problem of bending of a plate one may not prescribe more than two conditions on its contour; any third condition may bring about a contradiction in the statement of the problem, which is somewhat restricted by the adopted hypothesis of linear elements. By confining ourselves to two conditions, we have to meet the most important requirements imposed by the manner in which the plate is fixed or supported. Consider the fundamental cases encountered in design practice (Fig. 98).

1. *The left-hand edge is clamped.* Accordingly, we impose the conditions: for $x=0$, $w=0$, $\frac{\partial w}{\partial x}=0$, requiring that the deflection and the angle of inclination of the tangent to the middle plane at the support be absent.

2. *The back edge is simply supported.* Consequently, the deflection and the bending moment M_2 [formulas (10.12)] must be zero

at the support; thus

$$\text{for } y=0,\ w=0,\ \frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}=0. \qquad (10.23)$$

In these two cases no conditions have been imposed on shearing forces and twisting moments; but, being applied to the edge resting on the support, they will have some effect only on the magnitude of the supporting reactions of the plate and on the stress distribution near this edge.

3. *The right-hand edge is free.* This case presents the greatest difficulties. No restrictions are imposed here on the deflection; however, all three forces M_1, H, N_1 must be equal to zero and there

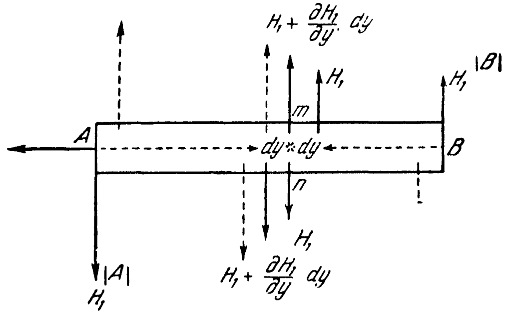

Fig. 100

is no reason to disregard any of them by treating them as secondary. Kirchhoff suggested the following way out of this difficulty. Let us suppose that the face $x=a$ is acted upon by distributed torsional couples. Their distribution is assumed to be nonuniform, in general, the magnitude of the moment of these couples per unit length of the face at a given point being denoted by H_1. The magnitude of the moment per length dy of the face is obviously equal to $H_1 dy$.

It will be noted that from the viewpoint of statics the distributed torsional couples are equivalent to a certain shearing force. Indeed, the couple with the moment $H_1 dy$ may be applied (Fig. 100) by means of two equal and opposite forces H_1 acting at the edges of an area of length dy.

The couple $\left(H_1+\frac{\partial H_1}{\partial y}dy\right)dy$ on the adjacent area of length dy may also be applied as two opposite forces $H_1+\frac{\partial H_1}{\partial y}dy$ with the arm dy. Having done this for all areas of the face concerned we see that the forces applied at points m and n on the boundary of the two areas reduce to a single force $\frac{\partial H_1}{\partial v}dy$ per length dy.

We observe, however, that there will remain two nonvanishing finite concentrated forces H_1^A and H_1^B at the edges A and B of the face.

Hence we conclude that the distributed torsional couples of intensity H_1 are statically equivalent to the distributed shearing force which has the intensity

$$\frac{\partial H_1}{\partial y} \tag{10.24}$$

and two concentrated shearing forces H_1^A and H_1^B at the ends, i.e. at the corners of the plate; if the contour is smooth and without corners, these concentrated forces are absent. Proceeding from this, Kirchhoff proposed that the three boundary conditions at the free edge be combined into two by equating to zero the bending moment M_1 and the shearing force N_1 and by adding to the latter the term (10.24), which reflects the influence of the twisting moment H. Now we arrive at the following two conditions at the free edge:

$$\left.\begin{aligned} M_1 &= 0, \\ N_1 + \frac{\partial H_1}{\partial y} &= 0 \end{aligned}\right\} \text{ or } \left.\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} &= 0, \\ -\left[\frac{\partial^3 w}{\partial x^3} + (2-\sigma)\frac{\partial^3 w}{\partial x\,\partial y^2}\right] &= 0. \end{aligned}\right\} \tag{10.25}$$

This form of the boundary conditions for a free edge is conventional.

There have now appeared some publications whose object is to eliminate the restrictions on the foregoing approximate theory of bending of plates imposed by Kirchhoff's hypothesis of linear elements; this hypothesis, as we see, makes it difficult to satisfy all the necessary boundary conditions. On the other hand, it is somewhat contradictory, since, taking into account the shearing stresses $Z_x = X_z$ and $Z_y = Y_z$, this theory eliminates the possibility of the pertinent shearing strains e_{xz} and e_{yz}; their consideration permits the refinement of the theory and of the respective boundary conditions [1]. Consider some special cases of bending of plates.

72. ELLIPTIC PLATE CLAMPED AT THE EDGE

Let us take the expression of deflection in the form

$$w = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2; \tag{10.26}$$

[1] E. Reissner, J. Math. and Phys., 1944, Vol. 23, pp. 184-191; L. Bolle Bull. techn. Suisse Romande, 1947, Nos. 11 et 25 oct.; B. F. Vlasov, Vestn Mosk. Univ., 1957, No. 2, pp. 25-34; Izv. AN SSSR, Otd. tekhn. nauk, 1958 No 12, pp. 124-127; Dokl. AN Azerb. SSR, 1957, Vol. 13, No. 9, pp. 955-959.

designating for the sake of brevity

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=U, \tag{10.27}$$

we have

$$w=cU^2. \tag{10.28}$$

We observe that the deflection is zero on the elliptic contour:

$$U=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0. \tag{10.29}$$

Consequently, we shall obtain the solution of the problem of bending of an elliptic plate if function (10.26) satisfies the basic equation (10.10′).

Taking the first derivatives of deflection (10.28)

$$\frac{\partial w}{\partial x}=2cU\frac{\partial U}{\partial x}, \quad \frac{\partial w}{\partial}=2cU\frac{\partial U}{\partial y},$$

we see that they vanish everywhere on the contour (10.29); hence we readily conclude that the middle plane remains horizontal on the contour; consequently, we have a plate clamped at the edge. Equation (10.10′) involves the fourth derivatives of deflection; one sees from expression (10.26) that they will all be constant numbers; it is evident from this that equation (10.10′) can be satisfied if $q=\text{const}$, i.e., if the load is continuous and uniform.

The calculation of the successive derivatives gives

$$\left.\begin{aligned}
&\frac{\partial w}{\partial x}=4\frac{c}{a^2}Ux; \quad \frac{\partial w}{\partial y}=4\frac{c}{b^2}Uy;\\
&\frac{\partial^2 w}{\partial x^2}=8\frac{c}{a^4}x^2+4\frac{c}{a^2}U; \quad \frac{\partial^2 w}{\partial x\,\partial y}=8\frac{c}{a^2b^2}xy;\\
&\frac{\partial^2 w}{\partial y^2}=8\frac{c}{b^4}y^2+4\frac{c}{b^2}U; \quad \frac{\partial^3 w}{\partial x^3}=24\frac{c}{a^4}x;\\
&\frac{\partial^3 w}{\partial x^2\,\partial y}=8\frac{c}{a^2b^2}y; \quad \frac{\partial^3 w}{\partial x\,\partial y^2}=8\frac{c}{a^2b^2}x; \quad \frac{\partial^3 w}{\partial y^3}=24\frac{c}{b^4}y;\\
&\frac{\partial^4 w}{\partial x^4}=24\frac{c}{a^4}; \quad \frac{\partial^4 w}{\partial x^2\,\partial y^2}=8\frac{c}{a^2b^2}; \quad \frac{\partial^4 w}{\partial y^4}=24\frac{c}{b^4}.
\end{aligned}\right\} \tag{10.30}$$

Substituting the values of the derivatives in equation (10.10′), we find the constant c, which is not yet determined:

$$c=\frac{q}{\left(\frac{24}{a^4}+\frac{16}{a^2b^2}+\frac{24}{b^4}\right)D}.$$

Introducing this value in (10.26), we get the equation of the deflected middle plane.

The reader is recommended to do the following exercises:

1. Find the maximum deflection of the plate at the centre.
2. Making use of formulas (10.12), (10.13), (10.14) and expressions of derivatives (10.30), find the general expressions of M_1, M_2, $H_1 = -H_2$, N_1 and N_2.
3. Calculate the maximum bending moments M_1 and M_2 at the ends of the axes of the contour (moments of clamping) and at the centre of the plate.
4. Setting $b=a$ (circular plate), find the quantities indicated in the problems above.

73. RECTANGULAR PLATE. NAVIER'S SOLUTION

Consider a plate simply supported at the edge and subjected to an arbitrary load. In this case, the solution of the basic equation (10.10′), i.e., the expression of the deflection w, has to be sought in the form of an infinite trigonometric series. Let us take it in the following form which, as may easily be seen, satisfies the supporting conditions on the four sides of the contour [1]:

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{10.31}$$

If we denote for the sake of brevity

$$A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = U_{mn}, \tag{10.32}$$

then series (10.31) will be represented as

$$\left.\begin{array}{l} \displaystyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} = \\ = U_{11} + U_{12} + U_{13} + \dots + U_{1k} + \dots + U_{1n} + U_{1n+1} + \dots \\ + U_{21} + U_{22} + U_{23} + \dots + U_{2k} + \dots + U_{2n} + \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + U_{m1} + U_{m2} + U_{m3} + \dots + U_{mk} + \dots + U_{mn} + \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{10.33}$$

each line of this expression is a series with respect to the subscript n, while each column is a series with respect to the subscript m.

---

[1] Indeed, on the contour, i.e., at $x=0$, $x=a$, $y=0$, $y=b$ we have $w=0$, $\frac{\partial^2 w}{\partial x^2} = 0$, $\frac{\partial^2 w}{\partial y^2} = 0$, and the conditions for the simply supported edge will be satisfied.

Differentiating expression (10.32) and substituting its derivatives for w in the left-hand member of equation (10.10′), we get

$$DA_{mn}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};$$

therefore, equation (10.10′) will be written as

$$D\pi^4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=q\,(x,\ y), \quad (10.34)$$

or more compactly:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=q\,(x,\ y), \quad (10.34a)$$

where

$$C_{mn}=D\pi^4A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2. \quad (10.35)$$

It remains to choose the coefficients of the series so as to satisfy equation (10.34a) identically over the whole area of the plate; in other words, according to (10.34a), our problem consists in expanding the function $q(x,\ y)$ in a trigonometric series in two variables x and y; this is carried out by the usual Euler's procedure for calculating the coefficients of a trigonometric series; since we have a double series, the procedure should be applied twice. First we multiply both members of equality (10.34a) by

$$\sin\frac{k\pi y}{b}\,dy,$$

where k is an integer, and integrate between the limits 0 and b:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x}{a}\int_0^b\sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}\,dy=\int_0^b q\,(x,\ y)\sin\frac{k\pi y}{b}\,dy. \quad (10.36)$$

The composition of this double sum is, of course, the same as in expression (10.33), but it is known that if $n\neq k$, then

$$\int_0^b\sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}\,dy=0;$$

if, however, $n=k$, then

$$\int_0^b\sin^2\frac{k\pi y}{b}\,dy=\frac{b}{2}. \quad (10.37)$$

Hence it is easy to conceive that in the sum of the left-hand member of (10.36) all terms vanish except those corresponding to the kth column of (10.33) in which m has all values 1, 2, 3, $\ldots, \infty$, and on the basis of (10.37) equality (10.36) becomes

$$\frac{b}{2}\sum_{m=1}^{\infty} C_{mk}\sin\frac{m\pi x}{a}=\int_0^b q(x, y)\frac{\sin k\pi y}{b}\,dy. \qquad (10.38)$$

Both sides of this equality are functions of x only. We now apply the above procedure to equality (10.38): we multiply both sides by

$$\sin\frac{i\pi x}{a}\,dx$$

and integrate from 0 to a; all terms except the one having the subscript $m=i$ vanish in the left-hand member; by analogy with (10.37) we have

$$\int_0^a \sin^2\frac{i\pi x}{a}\,dx=\frac{a}{2};$$

then from equality (10.38) we obtain

$$\frac{b}{2}\cdot\frac{a}{2}C_{ik}=\int_0^a\int_0^b q(x, y)\sin\frac{i\pi x}{a}\sin\frac{k\pi y}{b}\,dx\,dy.$$

Hence we find the coefficient C_{ik},

$$C_{ik}=\frac{4}{ab}\int\int q(x, y)\sin\frac{i\pi x}{a}\sin\frac{k\pi y}{b}\,dx\,dy,$$

and further, on the basis of (10.35), we write

$$A_{ik}=\frac{4}{D\pi^4\left(\frac{i^2}{a^2}+\frac{k^2}{b^2}\right)^2 ab}\int\int q(x, y)\sin\frac{i\pi x}{a}\sin\frac{k\pi y}{b}\,dx\,dy.$$

This is the general expression of any coefficient of series (10.31); replacing the symbols i and k again by m and n, we obtain

$$A_{mn}=\frac{4}{D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 ab}\int\int q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \qquad (10.39)$$

Given the distribution of load $q(x, y)$, it is possible to evaluate the integrals involved here and, substituting the values of coefficients A_{mn} in equation (10.31), to find the equation of the deflected middle plane. Consider two fundamental cases which are essential in design practice.

1. Let the load q be continuous and uniform over the whole area of the plate. Then, for calculating the coefficient A_{mn} given by equation (10.39), we have

$$\int\int q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy = q\int_0^a \sin\frac{m\pi x}{a}\,dx\int_0^b \sin\frac{n\pi y}{b}\,dy. \tag{10.40}$$

Consequently, the double integral has fallen into two simple integrals; we evaluate them:

$$\int_0^a \sin\frac{m\pi x}{a}\,dx = \frac{a}{m\pi}\left[-\cos\frac{m\pi x}{a}\right]_0^a = \frac{a}{m\pi}(1-\cos m\pi),$$

$$\int_0^b \sin\frac{k\pi y}{b}\,dy = \frac{b}{n\pi}(1-\cos n\pi).$$

For m and n odd these integrals are equal to $\frac{2a}{m\pi}$ and $\frac{2b}{n\pi}$, respectively; for m and n even they vanish. On this basis, integral (10.40) is different from zero only for m and n odd and then it is equal to

$$\frac{4qab}{\pi^2 mn}.$$

Substituting this value in (10.39), we get

$$A_{mn} = \frac{16q}{D\pi^6 mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2},$$

where $m=1, 3, 5, \ldots, \infty$; $n=1, 3, 5, \ldots, \infty$.

We observe that with increasing m and n the magnitude of the coefficient A_{mn} drops off sharply and series (10.31) converges rapidly; we obtain ultimately

$$w = \frac{16q}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}, \tag{10.41}$$

where $m=1, 3, 5, \ldots, \infty$; $n=1, 3, 5, \ldots, \infty$.

In order to find the maximum deflection at the centre of the plate we set $x=0.5a$; $y=0.5b$; then

$$f = \max w = \frac{16}{\pi^6 D}\sum\sum\frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} = \frac{16q}{\pi^6 D}\beta. \tag{10.42}$$

Here D should be replaced by its expression from (10.11). For calculation it is convenient to set $\frac{b}{a}=\mu$ and to represent the last formula as

$$f=\alpha\frac{qa^4}{Eh^3}.$$

The coefficient α is expressed by the double series [cf. formula (10.42)] and depends on the ratio of the sides μ (see Table).

$\mu=\frac{b}{a}$	$\max w=\alpha\frac{qa^4}{Eh^3}$	$\max M_1=\beta qa^2$	$\max M_2=\beta_1 qa^2$	$\max N_1=\gamma qa$	$\max N_2=\gamma_1 qb$	$\max\left(N_1+\frac{\partial H_1}{\partial y}\right)=\delta qa$	$\max\left(N_1+\frac{\partial H_2}{\partial x}\right)=\delta_1 qb$	$R=mqab$
μ	α	β	β_1	γ	γ_1	δ	δ_1	m
1.0	0.0443	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
1.1	0.0530	0.0553	0.0494	0.360	0.315	0.440	0.400	0.064
1.2	0.0616	0.0626	0.0501	0.380	0.294	0.455	0.377	0.062
1.3	0.0697	0.0693	0.0504	0.397	0.275	0.468	0.357	0.061
1.4	0.0770	0.0753	0.0506	0.411	0.258	0.478	0.337	0.059
1.5	0.0843	0.0812	0.0500	0.424	0.242	0.486	0.323	0.057
1.6	0.0906	0.0862	0.0493	0.435	0.229	0.491	0.303	0.054
1.7	0.0964	0.0908	0.0486	0.444	0.216	0.496	0.287	0.052
1.8	0.1017	0.0948	0.0479	0.452	0.205	0.499	0.273	0.050
1.9	0.1064	0.0985	0.0471	0.459	0.194	0.502	0.260	0.048
2.0	0.1106	0.1017	0.0464	0.465	0.185	0.503	0.248	0.046
3.0	0.1336	0.1189	0.0404	0.493	0.124	0.505	0.166	0.031
4.0	0.1400	0.1235	0.0384	0.498	0.093	0.502	0.125	0.024
5.0	0.1416	0.1246	0.0375	0.500	0.074	0.500	0.100	0.019
∞	0.1422	0.1250	0.0375	0.500	—	0.500	—	—

With the expression of the deflection (10.41) we can find, by equations (10.12), (10.13) and (10.14), the forces at any point of the plate, for instance:

$$M_1=\frac{16qa^2}{\pi^4}\sum\sum\frac{m^2+\sigma\frac{n^2}{\mu^2}}{mn\left(m^2+\frac{n^2}{\mu^2}\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$M_2=\frac{16qa^2}{\pi^4}\sum\sum\frac{\frac{n^2}{\mu^2}+\sigma m^2}{mn\left(m^2+\frac{n^2}{\mu^2}\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

Inasmuch as the edges are simply supported, the maximum moment occurs at the centre of the plate for $x=0.5a$, $y=0.5b$:

$$\max M_1 = \frac{16qa^2}{\pi^4} \sum \sum \frac{m^2 + c\dfrac{n^2}{\mu^2}}{mn\left(m^2+\dfrac{n^2}{\mu^2}\right)^2} (-1)^{\frac{m+n}{2}+1} = \beta qa^2,$$

$$\max M_2 = \frac{16qa^2}{\pi^4} \sum \sum \frac{\dfrac{n^2}{\mu^2} + \sigma m^2}{mn\left(m^2+\dfrac{n^2}{\mu^2}\right)^2} (-1)^{\frac{m+n}{2}+1} = \beta_1 qa^2.$$

The shearing forces and supporting reactions of the plate on the contour are calculated in the same manner. The results are presented in the table compiled by Academician B. G. Galerkin.
The quantities

$$\max\left(N_1 + \frac{\partial H_1}{\partial y}\right) = \delta aq; \quad \max\left(N_2 + \frac{\partial H_2}{\partial x}\right) = \delta_1 aq$$

represent the maximum unit supporting reactions at the middle of the sides of the contour. $R = mabq$ are the concentrated reactions at the corners of the plate; in calculation σ is taken to be 0.3.

2. A plate is bent by a concentrated force P applied at any point $x=c$, $y=d$. We replace the force P by a continuous load q distributed over an infinitesimal area $dx\,dy$:

$$P = q\,dx\,dy \quad \text{or} \quad q = \frac{P}{dx\,dy}. \tag{10.43}$$

Then, the function $q(x, y)$ in formula (10.39) is zero everywhere except at the point $x=c$, $y=d$, where it is equal to (10.43). In the integral of the right-hand member of equation (10.39) there remains only one nonvanishing element which corresponds to the point (c, d):

$$\int\int q(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,dx\,dy =$$
$$= \frac{P}{dx\,dy} \sin\frac{m\pi c}{a} \sin\frac{n\pi d}{b}\,dx\,dy = P \sin\frac{m\pi c}{a} \sin\frac{n\pi d}{b};$$

then, according to equation (10.39),

$$A_{mn} = \frac{4P\sin\dfrac{m\pi c}{a}\sin\dfrac{n\pi d}{b}}{D\pi^4 ab\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^2}.$$

From (10.31) we obtain the equation of the deflected middle plane

$$w = \frac{4P}{D\pi^4 ab} \sum \sum \frac{\sin \frac{m\pi c}{a} \sin \frac{n\pi d}{b}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (10.44)$$

74. RECTANGULAR PLATE. LÉVY'S SOLUTION

The method given by M. Lévy is more general than Navier's method; at the same time it is closely related to the solutions of L. N. G. Filon and C. Ribière of the plane problem for a rectangle

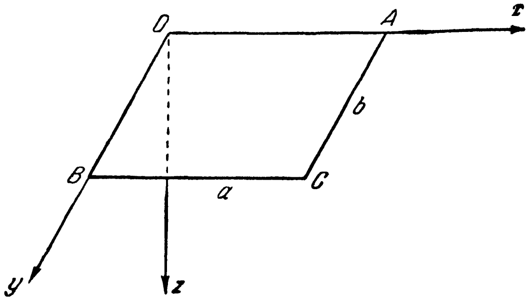

Fig. 101

presented in Sec. 44, which is due to the above-mentioned close relationship between the basic equations:

for the deflection of a plate

$$\nabla^2 \nabla^2 w = \frac{q(x, y)}{D} \quad (10.45)$$

and for the stress function in the plane problem

$$\nabla^2 \nabla^2 \varphi = 0.$$

Lévy's method is valid for all cases where (Fig. 101) two opposite edges of a rectangular plate, for instance OB and AC, are simply supported, while the other two, OA and BC, may be supported in any manner or be free. Bearing in mind these conditions, we seek the solution of equation (10.45) in the form

$$w = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a}. \quad (10.46)$$

Since

$$\frac{\partial^2 w}{\partial x^2} = -\frac{\pi^2}{a^2}\sum m^2 f_m(y)\sin\frac{m\pi x}{a},$$

we have

$$\left.\begin{array}{l} x=0 \\ x=a \end{array}\right\} \quad \frac{\partial^2 w}{\partial x^2} = 0.$$

i.e., the conditions at the simply supported edges OB and AC are satisfied. Substituting (10.46) in the left-hand member of equation (10.45), we obtain

$$\sum_{m=1}^{\infty}\left[f_m^{(\mathrm{IV})}(y) - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y) + \left(\frac{m\pi}{a}\right)^4 f_m(y)\right]\sin\frac{m\pi x}{a} = \frac{q(x, y)}{D}, \tag{10.47}$$

or more compactly

$$\sum_{m=1}^{\infty} D_m^*(y)\sin\frac{m\pi x}{a} = \frac{q(x, y)}{D}, \tag{10.48}$$

where

$$D_m^* = f_m^{(\mathrm{IV})}(y) - 2\left(\frac{m\pi}{a}\right)^2 f_m''(y) + \left(\frac{m\pi}{a}\right)^4 f_m(y). \tag{10.49}$$

The left-hand member represents a Fourier series in terms of sines; therefore, we must think of the right-hand member as being expanded in the same series

$$\frac{q(x, y)}{D} = \sum_{m=1}^{\infty} F_m(y)\sin\frac{m\pi x}{a},$$

where $F_m(y)$ are the coefficients in the series which are dependent on y since the prescribed load $q(x, y)$ is generally dependent on it; substituting this in the right-hand member of (10.48), we equate the coefficients of the respective terms on the left-hand and right-hand sides:

$$D_m^*(y) = F_m(y),$$

and this, according to (10.49), leads to an ordinary differential equation of the fourth order for finding the function $f_m(y)$, which is as yet unknown.

In practice, this process is reduced to the following formal manipulation: we multiply both members of (10.47) by $\sin\frac{n\pi x}{a}$ (where $n=1, 2, \ldots$) and integrate from zero to a; taking into

account that

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\,dx$$

is equal to zero for $m \neq n$ and to $\frac{a}{2}$ for $m=n$, we obtain the required differential equation

$$f_n^{(IV)}(y) - 2\left(\frac{n\pi}{a}\right)^2 f_n''(y) + \left(\frac{n\pi}{a}\right)^4 f_n(y) = \frac{2}{Da}\int_0^a q(x,\ y)\sin\frac{n\pi x}{a}\,dx. \tag{10.50}$$

Upon integration in the right-hand member of this equation, it will obviously become a function of y alone.

The general solution of equation (10.50) will be composed of the general solution of the homogeneous equation which coincides with equation (6.82) of Sec. 44, if we set in the latter

$$\lambda = \frac{n\pi}{a},$$

and of any particular solution of the complete equation (10.50); we denote this particular solution by

$$\bar{f}_n(y);$$

the general solution of the homogeneous equation can be written at once, on the basis of the arguments of Sec. 44, in the form of formula (6.88) by replacing m by n and l by a:

$$f_n^{(0)}(y) = A_n\cosh\frac{n\pi y}{a} + B_n\sinh\frac{n\pi y}{a} + C_n y\cosh\frac{n\pi y}{a} + D_n y\sinh\frac{n\pi y}{a}. \tag{10.51}$$

Therefore, the general solution of equation (10.50) is

$$f_n(y) = f_n^{(0)}(y) + \bar{f}_n(y) = A_n\cosh\frac{n\pi y}{a} +$$
$$+ B_n\sinh\frac{n\pi y}{a} + C_n y\cosh\frac{n\pi y}{a} + D_n y\sinh\frac{n\pi y}{a} + \bar{f}_n(y). \tag{10.52}$$

Introducing this value of $f_n(y)$ in (10.46), we find the solution of equation (10.45), which corresponds to the prescribed load $q(x,\ y)$ and satisfies the boundary conditions at the simply supported edges OB and AC. It remains to adjust the arbitrary constants

$$A_n,\quad B_n,\quad C_n,\quad D_n \tag{10.53}$$

so that every individual term of the sum (10.46) will satisfy the boundary conditions prescribed at the other two edges OA

and BC; the number of these conditions is always four, and they suffice to determine constants (10.53). Consider, for instance, the case where the edges OA and BC are also simply supported; then the boundary conditions are

$$\left.\begin{matrix} y=0 \\ y=b \end{matrix}\right\} \quad w=0, \quad \frac{\partial^2 w}{\partial y^2}=0, \tag{10.54}$$

or, according to (10.46), we have

$$\sum f_n(0)\sin\frac{n\pi x}{a}=0; \quad \sum f''_n(0)\sin\frac{n\pi x}{a}=0,$$

$$\sum f_n(b)\sin\frac{n\pi x}{a}=0; \quad \sum f''_n(b)\sin\frac{n\pi x}{a}=0,$$

or on account of the arbitrariness of x $(0\leqslant x\leqslant a)$ we get

$$\left.\begin{matrix} f_n(0)=0; & f''_n(0)=0; \\ f_n(b)=0; & f''_n(b)=0. \end{matrix}\right\} \tag{10.55}$$

Writing out these conditions by the use of (10.52), we obtain four equations:

$$\left.\begin{aligned} & A_n=-\bar{f}^{(0)}_n(0), \\ & A_n\left(\frac{n\pi}{a}\right)^2+D_n 2\frac{n\pi}{a}=-\bar{f}''_n(0), \\ & A_n\cosh\frac{n\pi b}{a}+B_n\sinh\frac{n\pi b}{a}+C_n b\cosh\frac{n\pi b}{a}+ \\ & \qquad\qquad +D_n b\sinh\frac{n\pi b}{a}=-\bar{f}_n(b), \\ & A_n\left(\frac{n\pi}{a}\right)^2\cosh\frac{n\pi b}{a}+B_n\left(\frac{n\pi}{a}\right)^2\sinh\frac{n\pi b}{a}+ \\ & \quad +C_n\left[2\frac{n\pi}{a}\sinh\frac{n\pi b}{a}+\left(\frac{n\pi}{a}\right)^2 b\cosh\frac{n\pi b}{a}\right]+ \\ & \quad +D_n\left[2\frac{n\pi}{a}\cosh\frac{n\pi b}{a}+\left(\frac{n\pi}{a}\right)^2 b\sinh\frac{n\pi b}{a}\right]=-\bar{f}''_n(b). \end{aligned}\right\} \tag{10.56}$$

After substituting in the right-hand members of these equations the appropriate values of the particular solution

$$\bar{f}_n(y), \tag{10.57}$$

we must solve them with respect to constants (10.53). The solution is greatly simplified if, taking advantage of the free choice of the particular solution (10.57), we construct it so that

$$\bar{f}_n(0)=0; \quad \bar{f}''_n(0)=0. \tag{10.58}$$

Then the first two equations of (10.56) give

$$A_n = D_n = 0. \tag{10.59}$$

Solving the other two equations, we get

$$\left.\begin{aligned} B_n &= \frac{1}{2\frac{n\pi}{a}\sinh^2\frac{n\pi b}{a}}\left\{b\cosh\frac{n\pi b}{a}\bar{f}_n''(b) - \right. \\ &\quad \left. - \left[2\frac{n\pi}{a}\sinh\frac{n\pi b}{a} + \left(\frac{n\pi}{a}\right)^2 b\cosh\frac{n\pi b}{a}\right]\bar{f}_n(b)\right\}, \\ C_n &= \frac{1}{2\frac{n\pi}{a}\sinh^2\frac{n\pi b}{a}}\left[\left(\frac{n\pi}{a}\right)^2\bar{f}_n(b) - \bar{f}_n''(b)\right]. \end{aligned}\right\} \tag{10.60}$$

The particular solution (10.57) which satisfies conditions (10.58) will be derived by Cauchy's method. For that, we first find from (10.51), by a suitable choice of constants (10.53), the particular solution of the homogeneous equation

$$D_n^*(y) = 0,$$

which satisfies the initial conditions

$$f_n(0) = f_n'(0) = f_n''(0) = 0; \quad f_n'''(0) = 1; \tag{10.61}$$

we denote it by $Y_n(y)$; it is given by

$$Y_n = \frac{a^2}{2(n\pi)^2}\left[y\cosh\frac{n\pi y}{a} - \frac{a}{n\pi}\sinh\frac{n\pi y}{a}\right]. \tag{10.62}$$

By differentiating, we find that this solution satisfies conditions (10.61). If, for brevity, we introduce in the right-hand member of (10.50) the notation

$$\frac{2}{Da}\int_0^a q(x, y)\sin\frac{n\pi x}{a}\,dx = \varphi_n(y), \tag{10.63}$$

then, following Cauchy's method, we find the required particular solution of equation (10.50) in the following form:

$$\bar{f}_n(y) = \int_0^y Y_n(y-\eta)\,\varphi_n(\eta)\,d\eta, \tag{10.64}$$

or in the expanded form

$$\begin{aligned} \bar{f}_n(y) = \frac{a}{D(n\pi)^2}\int_0^y \left\{\left[(y-\eta)\cosh\frac{n\pi(y-\eta)}{a} - \right.\right. \\ \left.\left. - \frac{a}{n\pi}\sinh\frac{n\pi(y-\eta)}{a}\right]\int_0^a q(x, \eta)\sin\frac{n\pi x}{a}\,dx\right\}d\eta. \end{aligned} \tag{10.64a}$$

Substituting the values (10.59), (10.60) and (10.64) in (10.52), we finally find the function $f_n(y)$, and, introducing it in (10.46), we obtain the solution for a rectangular plate simply supported along its contour and subjected to any prescribed load $q(x, y)$. The forces will be determined in the usual way according to formulas (10.12), (10.13) and (10.14).

This solution is more convenient for application than Navier's solution given in Sec. 73, because of the good convergence of the correspoding series. In a similar manner one can derive solutions for other boundary conditions at the edges *OA* and *BC*; it is only necessary to modify equations (10.56) accordingly for finding constants (10.53); the further procedure of solution holds.

Many problems concerned with the bending of plates were solved by Academician B. G. Galerkin. All these results are collected in his fundamental work *Thin Elastic Plates*, Leningrad-Moscow, Gosstrojizdat, 1933.

75. CIRCULAR PLATE

In the investigation of a circular plate it is most convenient to use the polar co-ordinates r and θ; therefore, all the basic equations of the bending of plates will be transformed to the polar co-ordinates according to the formulas

$$x = r\cos\theta; \quad y = r\sin\theta.$$

Inasmuch as the left-hand member of the basic equation (10.10′) is completely analogous in structure to the left-hand member of equation (IX) of Sec. 39 for the plane problem, we can use the computations of Sec. 47 for the transformation in question by replacing the function φ by w. Thus we immediately obtain

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2},$$

and the basic equation (10.10) becomes [cf. (IXp) in Sec. 47]

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = q(r, \theta). \tag{10.65}$$

Here we shall deal only with the case where the load is independent of θ, i.e., when it is distributed similarly in all directions from the centre of the plate. In this case the deflected middle surface must obviously be a surface of revolution, the deflection w does not depend on θ and equation (10.65) is simplified [cf.

equation (IX′p) in Sec. 47]:

$$D\left(\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^3}\frac{\partial w}{\partial r}\right)=q. \qquad (10.66)$$

The general solution of this equation is the sum of the general solution w of the equation without the last term

$$\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^3}\frac{\partial w}{\partial r}=0, \qquad (10.67)$$

and any particular solution of the complete equation (10.66), however, we already know the general solution of equation (10.67) [equation (7.5) of Sec. 48]

$$w=C_1\ln r+C_2r^2\ln r+C_3+C_4r^2; \qquad (10.68)$$

therefore, the general solution of (10.66) is written as

$$w=C_1\ln r+C_2r^2\ln r+C_3+C_4r^2+\overline{w}. \qquad (10.69)$$

The particular solution $\overline{w}$ can readily be found if the load q is constant (continuous uniform load over the whole area of the plate); the form of equation (10.66) itself reveals that one may assume $\overline{w}=cr^4$, where c is a number to be determined; substituting this value in equation (10.66), we easily find that $c=q/64D$ and, therefore,

$$\overline{w}=\frac{qr^4}{64D}.$$

Substituting this in equation (10.69), we obtain the general solution for a circular plate under a continuous uniform load:

$$w=C_1\ln r+C_2r^2\ln r+C_3+C_4r^2+\frac{qr^4}{64D}. \qquad (10.70)$$

To calculate moments let us use equations (10.12) and (10.14), when transformed to the polar co-ordinates. If the axis Ox is directed along the radius-vector r of a given point, we have

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial r^2}; \quad \frac{\partial^2 w}{\partial y^2}=\frac{1}{r}\frac{\partial w}{\partial r}; \quad \frac{\partial^2 w}{\partial x\,\partial y}=0.$$

Introducing this in equations (10.12) and (10.14), we find

$$\left.\begin{aligned} M_1&=-D\left(\frac{\partial^2 w}{\partial r^2}+\frac{\sigma}{r}\frac{\partial w}{\partial r}\right),\\ M_2&=-D\left(\frac{1}{r}\frac{\partial w}{\partial r}+\sigma\frac{\partial^2 w}{\partial r^2}\right),\\ H_1&=H_2=0. \end{aligned}\right\} \qquad (10.71)$$

Here M_1 is the bending moment over the area normal to the radius, M_2 is the moment over the radial section; the twisting

moments over the basic sections under consideration are absent, as expected, because of the symmetry of the plate and loading.

Consider several particular modes of fixing a plate on its contour. If we take a solid plate without a circular cutout around its centre, we have to drop the first two terms in the general solution (10.70); otherwise, w, $\frac{\partial w}{\partial r}$ and $\frac{\partial^2 w}{\partial r^2}$ become infinite at the centre of the plate, i.e., the deflection and curvature of the plate will be equal to infinity. Hence we obtain

$$w = C_3 + C_4 r^2 + \frac{qr^4}{64D}. \tag{10.72}$$

The remaining two arbitrary constants can be determined from the fixing conditions.

(a) If the plate of radius a is clamped along its contour, we obtain the following conditions on the contour: for $r=a$ $w=0$, $\partial w/\partial r=0$. Determining C_3 and C_4 from these conditions and substituting their values in equation (10.72), we get

$$w = \frac{q}{64D}(a^2 - r^2)^2. \tag{10.73}$$

Further, according to equations (10.71), we find

$$\left.\begin{aligned} M_1 &= \frac{q}{16}[a^2(1+\sigma) - r^2(3+\sigma)], \\ M_2 &= \frac{q}{16}[a^2(1+\sigma) - r^2(1+3\sigma)]. \end{aligned}\right\} \tag{10.74}$$

The reader is recommended to verify these formulas and to find max M on the contour of the plate and at its centre, as well as to find the deflection at the centre. All these results are to be compared with the results of Sec. 72 in the case $b=a$.

(b) If the plate is simply supported along its contour, the conditions on the contour for $r=a$ are

$$w=0, \quad M_1 = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\sigma}{r}\frac{\partial w}{\partial r}\right) = 0;$$

determining C_3 and C_4 from these conditions, we obtain from equation (10.72)

$$w = \frac{q}{64D}\left[(a^2 - r^2)^2 + \frac{4a^2(a^2 - r^2)}{1+\sigma}\right]. \tag{10.75}$$

Further, we find, according to equations (10.71),

$$\left.\begin{aligned} M_1 &= \frac{q}{16}[a^2(3+\sigma) - r^2(3+\sigma)], \\ M_2 &= \frac{q}{16}[a^2(3+\sigma) - r^2(1+3\sigma)]. \end{aligned}\right\} \tag{10.76}$$

The reader is recommended to calculate max M and the deflection at the centre of the plate.

(c) Consider also the case of a plate with a simply supported edge in the absence of load ($q=0$) but subjected to a uniformly distributed moment $M_1=M$ along the contour. In this case, from equation (10.72), we obtain

$$w=C_3+C_4r^2.$$

The constants have to be determined from the conditions for $r=a$

$$w=0, \quad -D\left(\frac{\partial^2 w}{\partial r^2}+\frac{\sigma}{r}\frac{\partial w}{\partial r}\right)=M.$$

We readily find that

$$w=\frac{M}{2D(1+\sigma)}(a^2-r^2). \tag{10.77}$$

This solution appears to be necessary in the design of the end plates of circular cylindrical reservoirs. Here we encounter a case of pure bending of a plate.

In conclusion we present the results which are obtained in the case of a plate carrying a centrally placed concentrated force P. If the plate is clamped along its contour, we have

$$w=\frac{P}{8\pi D}\left[\frac{1}{2}(a^2-r^2)-r^2\ln\frac{a}{r}\right]. \tag{10.78}$$

If the plate is simply supported, we have

$$w=\frac{P}{8\pi D}\left[\frac{1}{2}\frac{3+\sigma}{1+\sigma}(a^2-r^2)-r^2\ln\frac{a}{r}\right]. \tag{10.79}$$

Calculation of moments according to equations (10.71) offers no difficulty.

Combining these solutions with those given above and equating to zero the deflection at the centre, one can solve the problem of a plate supported along the contour and at the centre and subjected to a continuous uniform load.

76. MEMBRANE ANALOGY. MARCUS'S METHOD

Let us indicate a new form assumed by the basic equations of a plate in introducing the Laplacian operator on the deflection

$$\nabla^2 w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$

and the so-called reduced sum of the moments [see formulas (10.12)]

$$\mathfrak{M}=\frac{M_1+M_2}{1+\sigma}=-D\nabla^2 w. \tag{10.80}$$

This leads to the differential equation

$$D\nabla^2 w = -\mathfrak{M}. \tag{10.81}$$

The expressions of shearing forces (10.13) can be written down as

$$\left.\begin{aligned} N_1 &= -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial \mathfrak{W}}{\partial x}, \\ N_2 &= -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial \mathfrak{M}}{\partial y}. \end{aligned}\right\} \tag{10.82}$$

These relationships are analogous to the known relationship for a straight rod

$$Q=\frac{dM}{dx}.$$

Next, equation (10.10) on the basis of (10.81) is rewritten as

$$D\nabla^2\left(-\frac{\mathfrak{W}}{D}\right)=q \quad \text{or} \quad \nabla^2\mathfrak{M}=-q. \tag{10.83}$$

For a straight rod there is a similar relation

$$\frac{d^2 M}{dx^2}=-q.$$

We shall now consider, parallel with the plate, a flexible inextensible membrane (film) clamped on the same contour as the plate in question and subjected to the same load $q(x, y)$; the differential equation of the surface of the membrane was obtained in Sec. 54 [see equation (8.71)]:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-\frac{q}{H}, \tag{10.84}$$

or more compactly

$$\nabla^2 z=-\frac{q}{H}. \tag{10.84a}$$

For a funicular curve we have a similar equation:

$$\frac{d^2 z}{\partial x^2}=-\frac{q}{H},$$

where H is the pole distance.

Let us now compare equations (10.81), (10.83) and (1084a):

$$\nabla^2 w=-\frac{\mathfrak{W}}{D}, \tag{10.81}$$

$$\nabla^2\mathfrak{M}=-\frac{q}{1}, \tag{10.83}$$

$$\nabla^2 z=-\frac{q}{H}. \tag{10.84a}$$

Hence we draw the following conclusions similar to those based on the analogy between the elastic line and the funicular curve.

1. If we set $H=1$, then $z=\mathfrak{M}$; consequently, when the membrane is subjected to tension equal to unity, its sagged surface under the load q gives the diagram of the reduced sum of the moments $\mathfrak{M}$ due to this load. (Compare with the M diagram as a funicular curve or funicular polygon for the given load.)

2. If we set $H=D$, $q=\mathfrak{M}$, then $z=w$; consequently, when the membrane is subjected to tension equal to the flexural rigidity D, the sagged surface of the membrane under the load in the form of the $\mathfrak{M}$ diagram gives the deflected surface of the plate. (Compare with the elastic line as a funicular curve due to the loading by the M diagram.)

By using this analogy between plate and membrane, H. Marcus devised his method of analysis of a plate; he replaces a membrane by a network (cf. the approximate replacement of a funicular curve by a funicular polygon) and thereby transforms its differential equation (10.84a) into a finite-difference equation; proceeding in this way, he replaces the integration of the differential equation (10.84a) by the solution of a system of equations of the first degree [1].

[1] See H. Marcus, *Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten,* Berlin, Springer, 1924.

XI

Variational Methods of the Theory of Elasticity

77. VARIATIONAL PRINCIPLES OF THE THEORY OF ELASTICITY. FUNDAMENTAL INTEGRAL IDENTITY

By variational principles of the theory of elasticity are meant certain basic theorems expressed in the form of integral equalities connecting stresses, strains and displacements throughout the volume of a body and based on the properties of the work of elastic forces. The variational principles are of practical interest in the sense that they form the basis of the methods which permit the effective solution of problems in many cases where the classical approach to the integration of the basic equations of the theory of elasticity presents difficulties which have as yet not been surmounted. In this section we shall fix our attention on an integral transformation which enables us to simplify further computations.

Consider a certain state of equilibrium or of small elastic vibrations of a body, which is characterised by the stresses

$$X_x, \quad Y_y, \quad Z_z, \quad Y_z, \quad Z_x, \quad X_y$$

and the displacements

$$u, \quad v, \quad w$$

and which satisfies the basic equations

$$\left.\begin{aligned}
&\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+X\rho=\rho\frac{\partial^2 u}{\partial t^2},\\
&\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}+Y\rho=\rho\frac{\partial^2 v}{\partial t^2},\\
&\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}+Z\rho=\rho\frac{\partial^2 w}{\partial t^2},
\end{aligned}\right\} \tag{11.1}$$

with external forces acting on the surface of the body:

$$\left.\begin{aligned}
X_v&=X_x\cos(vx)+X_y\cos(vy)+X_z\cos(vz),\\
Y_v&=Y_x\cos(vx)+Y_y\cos(vy)+Y_z\cos(vz),\\
Z_v&=Z_x\cos(vx)+Z_y\cos(vy)+Z_z\cos(vz).
\end{aligned}\right\} \tag{11.2}$$

At the same time consider three functions

$$u'(x, y, z, t); \quad v'(x, y, z, t); \quad w'(x, y, z, t), \tag{11.3}$$

which are continuous throughout the volume of the body and have continuous partial derivatives of the first and second order with respect to x, y, z and t in the same region. Let us regard them as displacements produced in the body under the action of some forces which are of no interest as yet. Because of displacements (11.3) there are induced strains

$$\left.\begin{aligned} e'_{xx} &= \frac{\partial u'}{\partial x}; \quad & e'_{yz} &= \frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}; \\ e'_{yy} &= \frac{\partial v'}{\partial y}; \quad & e'_{zx} &= \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}; \\ e'_{zz} &= \frac{\partial w'}{\partial z}; \quad & e'_{xy} &= \frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}. \end{aligned}\right\} \tag{11.4}$$

Let us form the following integral extended over the surface of the body

$$I = \int\limits_{(S)} (X_v u' + Y_v v' + Z_v w')\, dS. \tag{11.5}$$

Substituting here the values of X_v, Y_v, Z_v from (11.2) and collecting the coefficients of cosines, we get

$$I = \int\limits_{(S)} [P \cos(vx) + Q \cos(vy) + R \cos(vz)]\, dS, \tag{11.6}$$

where

$$\begin{aligned} P &= X_x u' + Y_x v' + Z_x w', \\ Q &= X_y u' + Y_y v' + Z_y w', \\ R &= X_z u' + Y_z v' + Z_z w'. \end{aligned}$$

Next, as in Sec. 37, we transform this surface integral into a volume integral

$$I = \int\limits_{(\tau)} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] d\tau, \tag{11.7}$$

where

$$\left.\begin{aligned} \frac{\partial P}{\partial x} &= \frac{\partial X_x}{\partial x} u' + \frac{\partial Y_x}{\partial x} v' + \frac{\partial Z_x}{\partial x} w' + X_x \frac{\partial u'}{\partial x} + Y_x \frac{\partial v'}{\partial x} + Z_x \frac{\partial w'}{\partial x}, \\ \frac{\partial Q}{\partial y} &= \frac{\partial X_y}{\partial y} u' + \frac{\partial Y_y}{\partial y} v' + \frac{\partial Z_y}{\partial y} w' + X_y \frac{\partial u'}{\partial y} + Y_y \frac{\partial v'}{\partial y} + Z_y \frac{\partial w'}{\partial y}, \\ \frac{\partial R}{\partial z} &= \frac{\partial X_z}{\partial z} u' + \frac{\partial Y_z}{\partial z} v' + \frac{\partial Z_z}{\partial z} w' + X_z \frac{\partial u'}{\partial z} + Y_z \frac{\partial v'}{\partial z} + Z_z \frac{\partial w'}{\partial z}. \end{aligned}\right\} \tag{11.8}$$

For substitution in (11.7), we add up these equalities by members, making use of equations (11.1) in adding up the first three columns on the right-hand side, and of equalities (11.4) in adding up the remaining columns. Hence we find.

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\rho\left(\frac{\partial^2 u}{\partial t^2}-X\right)u'+\rho\left(\frac{\partial^2 v}{\partial t^2}-Y\right)v'+$$

$$+\rho\left(\frac{\partial^2 w}{\partial t^2}-Z\right)w'+X_x e'_{xx}+Y_y e'_{yy}+Z_z e'_{zz}+Y_z e'_{yz}+Z_x e'_{zx}+X_y e'_{xy}. \tag{11.9}$$

We introduce the value (11.9) in (11.7) and replace I in the left-hand member of the latter by its expression (11.5); we then obtain

$$\int\limits_{(S)}[X_\nu u'+Y_\nu v'+Z_\nu w']\,dS+\int\limits_{(\tau)}\rho\left[\left(X-\frac{\partial^2 u}{\partial t^2}\right)u'+\left(Y-\frac{\partial^2 v}{\partial t^2}\right)v'+\right.$$

$$\left.+\left(Z-\frac{\partial^2 w}{\partial t^2}\right)w'\right]d\tau=\int\limits_{(\tau)}(X_x e'_{xx}+Y_y e'_{yy}+Z_z e'_{zz}+$$

$$+Y_z e'_{yz}+Z_x e'_{zx}+X_y e'_{xy})\,d\tau. \tag{11.10}$$

This equality results from formal computations; nevertheless, attaching one or another meaning, on the one hand, to the state of stress

$$X_x,\ Y_y,\ Z_z,\ Y_z,\ Z_x,\ X_y \tag{11.11a}$$

and to the respective displacements

$$u,\ v,\ w \tag{11.11b}$$

and, on the other hand, to the displacements

$$u',\ v',\ w' \tag{11.12}$$

we obtain all the general theorems of the theory of elasticity.

As one of the examples, we assume that the state of stress (11.11a) corresponds to the equilibrium of a body (i.e., $\rho\frac{\partial^2 u}{\partial t^2}=\rho\frac{\partial^2 v}{\partial t^2}=\rho\frac{\partial^2 w}{\partial t^2}=0$) and that the actual displacements u, v, w are taken as u', v', w'; then equality (11.10) is transformed into (5.62) and yields Clapeyron's theorem.

78. LAGRANGE'S VARIATIONAL EQUATION

Let the state of stress (11.11a) and displacements (11.11b) correspond to certain small elastic vibrations of a body (and, in particular, to the equilibrium state of a body). As u', v' and w'

we take arbitrary variations of the displacements u, v, w which do not violate the geometrical constraints[1] imposed on these displacements, and we denote

$$u' = \delta u; \quad v' = \delta v; \quad w' = \delta w.$$

Equation (11.10) assumes the following form:

$$\int\limits_{(S)} (X_v \delta u + Y_v \delta v + Z_v \delta w) dS + \int\limits_{(\tau)} \rho \left[\left(X - \frac{\partial^2 u}{\partial t^2} \right) \delta u + \right.$$
$$\left. + \left(Y - \frac{\partial^2 v}{\partial t^2} \right) \delta v + \left(Z - \frac{\partial^2 w}{\partial t^2} \right) \delta w \right] d\tau = \int\limits_{(\tau)} (X_x \delta e_{xx} + Y_y \delta e_{yy} +$$
$$+ Z_z \delta e_{zz} + Y_z \delta e_{yz} + Z_x \delta e_{zx} + X_y \delta e_{xy}) \, d\tau. \quad (11.13)$$

In the right-hand member it is taken into account that

$$e'_{xx} = \frac{\partial u'}{\partial x} = \frac{\partial}{\partial x} (\delta u) = \delta \left(\frac{\partial u}{\partial x} \right) = \delta e_{xx}$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$e'_{xy} = \frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y} = \frac{\partial}{\partial x} (\delta v) + \frac{\partial}{\partial y} (\delta u) = \delta \left(\frac{\partial v}{\partial x} \right) + \delta \left(\frac{\partial u}{\partial y} \right) = \delta e_{xy}$$

$$\cdot \; \cdot ,$$

i.e., δe_{xx}, δe_{yy}, ... represent the variations of the strain components corresponding to the state of stress (11.11a). The integrand in the right-hand member of (11.13) will be transformed by using formula (3.25):

$$X_x \delta e_{xx} + Y_y \delta e_{yy} + \ldots = \frac{\partial W}{\partial e_{xx}} \delta e_{xx} + \frac{\partial W}{\partial e_{yy}} \delta e_{yy} + \ldots = \delta W. \quad (11.13a)$$

---

[1] The examples of geometrical constraints are:

1. If a body rests on immovable supports at some points, the following conditions must be imposed on the variations:

$$\delta u = \delta v = \delta w = 0.$$

2. If an element dx at a given point is restrained from rotating in the planes Oxy and Oxz, then [cf. formulas (2.3), (2.4), (2.8)]

$$\alpha_{yx} = \frac{\partial v}{\partial x} = 0; \quad \alpha_{zx} = \frac{\partial w}{\partial x} = 0,$$

and hence the conditions follow which are imposed on the variations at this point:

$$\delta \left(\frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial x} (\delta v) = 0; \quad \delta \left(\frac{\partial w}{\partial x} \right) = \frac{\partial}{\partial x} (\delta w) = 0.$$

Here δW is the variation of the elastic energy W resulting from the variation of the displacements δu, δv, δw. Then

$$\int_{(\tau)} (X_x \delta e_{xx} + Y_y \delta e_{yy} + \ldots) d\tau = \int_{(\tau)} \delta W d\tau = \delta \int_{(\tau)} W d\tau$$

represents the variation of the elastic energy in the whole body. Equality (11.13) yields the variational equation

$$\int_{(S)} (X_v \delta u + Y_v \delta v + Z_v \delta w) dS + \int_{(\tau)} \rho \left[\left(X - \frac{\partial^2 u}{\partial t^2}\right) \delta u + \right.$$
$$\left. + \left(Y - \frac{\partial^2 v}{\partial t^2}\right) \delta v + \left(Z - \frac{\partial^2 w}{\partial t^2}\right) \delta w\right] d\tau = \delta \int_{(\tau)} W d\tau, \quad (11.14)$$

which formulates D'Alembert's general principle in relation to an elastic body.

We deal very frequently with problems of equilibrium of an elastic body; in this case accelerations are equal to zero and equation (11.14) formulates Lagrangé's principle, i.e., the principle of virtual displacements:

$$\int_{(S)} (X_v \delta u + Y_v \delta v + Z_v \delta w) dS + \int_{(\tau)} \rho (X \delta u +$$
$$+ Y \delta v + Z \delta w) d\tau = \delta \int_{(\tau)} W d\tau. \quad (11.15)$$

It should be emphasised that in forming equations (11.14) and (11.15) we vary the displacements u, v, w and the corresponding strains, while the external forces

$$X_v, \ Y_v, \ Z_v; \quad X\rho, \ Y\rho, \ Z\rho,$$

the stresses

$$X_x, \ Y_y, \ Z_z, \quad Y_z, \ Z_x, \ X_y$$

and the inertia forces

$$\rho \frac{\partial^2 u}{\partial t^2}, \quad \rho \frac{\partial^2 v}{\partial t^2}, \quad \rho \frac{\partial^2 w}{\partial t^2}$$

are not varied. Bearing this in mind, we put the variation sign before the integral signs in the left-hand member of equation (11.15) and write it down as

$$\delta \left[\int_{(S)} (X_v u + Y_v v + Z_{vw}) dS + \int_{(\tau)} \rho (Xu + Yv + Zw) d\tau \right] = \delta U,$$

where

$$U = \int_{(S)} (X_v u + Y_v v + Z_v w) dS + \int_{(\tau)} \rho (Xu + Yv + Zw) d\tau.$$

U is the work done by all external forces applied to the body on virtual displacements. But this is not the work A actually done by the external forces: the equilibrium will be attained only when the external forces increase infinitely slowly from zero to their final values (producing no accelerations); then, according to Clapeyron's theorem, $U=2A$ and equation (11.15) can be written in either of the following two forms:

$$\delta\left[U-\int_{(\tau)} W\,d\tau\right]=0, \tag{11.16a}$$

$$\delta\left[2A-\int_{(\tau)} W\,d\tau\right]=0. \tag{11.16b}$$

The quantity

$$F=U-\int_{(\tau)} W\,d\tau$$

is the force function and $\Pi=-F$ is the potential energy of the system of external and internal forces. Equation (11.16a) gives the necessary condition for a maximum or minimum of the potential energy of the system if an elastic body is in equilibrium. On the basis of Lejeune-Dirichlet's criterion [1] the potential energy has a minimum value if the equilibrium is stable. Instability of elastic equilibrium may occur, as is known, in the case of thin bodies: bars, plates and shells.

If displacements are prescribed on the surface of a body (i.e., if we have the second problem of the theory of elasticity), their variations should be equal to zero; if, at the same time, the body forces are absent, the variation of the work done by external forces is zero:

$$\delta U=0,$$

and equation (11.16a) becomes

$$\delta\int_{(\tau)} W\,d\tau=0. \tag{11.17}$$

It formulates the theorem of minimum elastic energy: the actual state of stress and strain is that for which the total elastic energy of the body has a minimum value. We shall call this theorem the *first theorem of minimum* and point out once again that the displacements and strains corresponding to the state of equilibrium are varied while the stresses are not; in fact, according to (11.13a)

$$\delta W=X_x\,\delta e_{xx}+Y_y\,\delta e_{yy}+Z_z\,\delta e_{zz}+Y_z\,\delta e_{yz}+Z_x\,\delta e_{zx}+X_y\,\delta e_{xy};$$

[1] It is given in the courses on theoretical mechanics.

in forming the force function

$$F = U - \int_{(\tau)} W\, d\tau, \tag{11.18}$$

whose minimum is to be sought according to (11.16a), the elastic energy W is calculated as the work of forces

$$\frac{1}{2}(X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + Y_z e_{yz} + Z_x e_{zx} + X_y e_{xy}), \tag{11.19}$$

and then stresses are expressed in terms of strains by the generalised Hooke's law; thus, W is represented as a function of strain components.

However, in investigating the bending of bars, plates and shells of small thickness the hypotheses of plane sections and linear elements introduced there permit the elastic energy to be calculated as the work of the bending and twisting moments and shearing forces. For instance, in the bending of a straight bar we isolate its element by two adjacent sections; then, neglecting the work done by the shearing force, we have the energy of bending of this element

$$\frac{1}{2} M\, d\varphi = \frac{1}{2} M \frac{dx}{\rho},$$

where $\frac{1}{\rho}$ is the curvature of the axis of the bar.

The relation between moment and deformation yields

$$M = \frac{EJ}{\rho}.$$

The energy of bending per unit length of the bar ($dx = 1$) is

$$W = \frac{EJ}{2}\left(\frac{1}{\rho}\right)^2.$$

But because of the simple approximate relation between deformation and deflection of the bar v

$$\frac{1}{\rho} = \frac{d^2 v}{dx^2},$$

we are in a position to express the elastic energy as a function of the displacement v:

$$W = \frac{1}{2} EJ \left(\frac{d^2 v}{dx^2}\right)^2.$$

For example, for a beam under a distributed load of intensity $q(x)$ the work of external forces in equation (11.16a) is

$$U = \int_0^l q(x)\, v(x)\, dx,$$

and the equation becomes

$$\delta\left[\int_0^l q(x)\,v(x)\,dx - \frac{EJ}{2}\int_0^l \left(\frac{d^2v}{dx^2}\right)^2 dx\right] = 0.$$

Thus, the problem reduces to the determination of such a function $v(x)$ that minimises the integral

$$F = \int_0^l \left[q(x)\,v(x) - \frac{EJ}{2}\left(\frac{d^2v}{dx^2}\right)^2\right] dx. \qquad (11.20)$$

In the case of the bending of a moderately thick plate we can also calculate the elastic energy as a function of the deflection w by using the results of Chapter X; on the basis of Clapeyron's theorem we obtain per unit area of the middle surface

$$\frac{1}{2}\left(M_1\frac{1}{\rho_1} + M_2\frac{1}{\rho_2} + H\tau\right);$$

here

$$\left.\begin{aligned}\frac{1}{\rho_1} &\approx \frac{\partial^2 w}{\partial y^2},\\ \frac{1}{\rho_2} &\approx \frac{\partial^2 w}{\partial y^2},\end{aligned}\right\}\ \text{are the curvatures of the middle surface,}$$

$$\tau = -2\frac{\partial w}{\partial x\,\partial y}\ \text{is the torsion.}$$

Substituting these values and the values of moments (10.12) and (10.14) in the preceding expression, we obtain the elastic energy of the isolated element:

$$dW = \frac{1}{2}D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\sigma\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \right.$$
$$\left. + 2(1-\sigma)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right] dx\,dy, \qquad (11.21')$$

integrating it over the whole of the middle surface we find the total energy of the plate. It does not involve the work of the shearing forces N_1 and N_2, since the corresponding shearing strains e_{xz} and e_{yz}, according to the hypothesis of linear elements, are zero. By using expression (11.21′), we form equation (11.16a) for a plate bent by a load $q(x, y)$. It requires the value of the integral

$$F = \int\int\left\{q(x, y)\,w(x, y) - \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \right.\right.$$
$$\left.\left. + 2\sigma\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\sigma)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy \qquad (11.21)$$

extended over the whole middle surface of the plate to be a minimum.

The problems of finding functions which minimise a definite integral are treated in the calculus of variations. Here we shall dwell only on the so-called direct method of the calculus of variations which was first introduced by W. Ritz in solving the problem of bending and vibrations of a plate, in relation to integral (11.21).

79. RITZ-TIMOSHENKO METHOD

The essence of this method is that the form of unknown functions u, v, w is prescribed in such a manner that the boundary conditions for them (the conditions on the surface of a body) are satisfied. At the same time these functions should contain a sufficient number of arbitrary parameters by varying which we can vary the functions themselves without violating the geometrical constraints imposed on the body (i.e., constraints preventing displacements).

In the case of the bending of a bar [formula (11.20)], the deflection is prescribed in the form

$$v(x) = c_1 v_1(x) + c_2 v_2(x) + \dots + c_n v_n(x), \qquad (11.22)$$

where $c_1, c_2, \dots, c_n$ are arbitrary parameters, and $v_1(x)$, $v_2(x)$, $\dots$, $v_n(x)$ are functions which are chosen arbitrarily, in general, but satisfy the conditions imposed on displacements. For instance, in the case of a cantilever fixed at the left-hand end ($x=0$) all functions must satisfy the conditions:

$$\text{for } x=0 \quad v_k(x) = 0, \quad \frac{dv_k}{dx} = 0. \qquad (11.23)$$

In the case of a beam simply supported at the ends ($x=0$ and $x=l$), the conditions will be:

$$\text{for } x=0 \text{ and } x=l \quad v_k(x) = 0. \qquad (11.24)$$

These *geometrical* conditions are alone obligatory; the satisfaction of statical conditions is but desirable. For instance, in the case of a cantilever subjected to a distributed load only there must be:

$$\left.\begin{aligned} &\text{for } x=l \quad \frac{d^2 v_k}{dx^2} = 0; \\ &\text{for } x=l \quad \frac{d^3 v_k}{dx^3} = 0. \end{aligned}\right\} \qquad (11.25)$$

The satisfaction of these conditions is not necessary, generally speaking, but they determine a more or less successful choice of functions $v_k(x)$, and the success of the application of the method itself in a given problem rests entirely on this choice. For example, if we assume for a simply supported beam

$$v(x) = c_1 x(l-x) + c_2 x^2 (l-x)^2 + \ldots + c_m x^m (l-x)^m. \quad (11.26)$$

the geometrical conditions (11.24) will be satisfied, while the statical conditions are not satisfied because the bending moments at the ends do not vanish; this circumstance, however, is alleviated if a sufficient number of terms is taken in formula (11.26). Assuming

$$v(x) = c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + \ldots + c_m \sin\frac{m\pi x}{l}, \quad (11.27)$$

we satisfy both the geometrical and statical conditions; therefore, we obtain a good result even for a small number of terms in formula (11.27).

To illustrate the procedure of solution we shall carry it through in the case of a uniform load. The work done by the load is

$$\int_0^l qv(x)\,dx = q\int_0^l \sum c_m \sin\frac{m\pi x}{l}\,dx =$$

$$= q\sum c_m \int_0^l \sin\frac{m\pi x}{l}\,dx = \frac{ql}{\pi}\sum \frac{c_m}{m}(1-\cos m\pi).$$

The terms of this sum vanish for m even; therefore,

$$\int_0^l qv(x)\,dx = \frac{2ql}{\pi}\sum c_m \frac{1}{m}; \quad m = 1, 3, 5, \ldots.$$

The elastic energy is

$$\frac{EJ}{2}\int_0^l \left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EJ}{2}\int_0^l \left[\sum\left(-\frac{\pi^2}{l^2} c_m m^2 \sin\frac{m\pi x}{l}\right)\right]^2 dx =$$

$$= \frac{EJ\pi^4}{2l^4}\sum c_m^2 m^4 \frac{l}{2} = \frac{\pi^4 EJ}{4l^3}\sum m^4 c_m^2.$$

The force function is

$$F = \int_0^l qv(x)\,dx - \frac{EJ}{2}\int_0^l \left(\frac{d^2v}{dx^2}\right)^2 dx =$$

$$= \frac{2ql}{\pi}\sum\frac{c_m}{m} - \frac{EJ\pi^4}{4l^3}\sum m^4 c_m^2; \quad (11.28)$$

it is a polynomial of the second degree in the coefficients c_m of function (11.27). In order to obtain a minimum of this function we equate to zero its partial derivatives with respect to c_m:

$$\frac{\partial F}{\partial c_m}=\frac{2ql}{\pi m}-\frac{EJ\pi^4}{4l^3}2m^4c_m=0.$$

Hence

$$c_m=\frac{4ql^4}{\pi^5 m^5 EJ};$$

$$v=\frac{4ql^4}{\pi^5 EJ}\left(\sin\frac{\pi x}{l}+\frac{1}{3^5}\sin\frac{3\pi x}{l}+\frac{1}{5^5}\sin\frac{5\pi x}{l}+\ldots\right). \quad (11.29)$$

The maximum deflection (at $x=l/2$) is

$$f=v_{max}=\frac{4ql^4}{\pi^5 EJ}\left(1-\frac{1}{3^5}+\frac{1}{5^5}-\ldots\right).$$

If we retain only the first terms in formula (11.27), then, taking into account that the error in an alternating series is less than the first omitted term, we see that this error is less than

$$\frac{1}{3^5}=\frac{1}{243}=0.0041=0.4\%$$

and

$$f=\frac{4ql^4}{\pi^5 EJ}=\frac{5ql^4}{382.5EJ}$$

instead of the exact value $F=\frac{5ql^4}{384EJ}$.

In the case of the bending of a rectangular plate having the sides a and b long and simply supported along the contour, its deflection can be expressed as

$$w=\sum\sum C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.^{1} \quad (11.30)$$

Substituting this value in the expression of the elastic energy (11.21′) and carrying out integration, we obtain

$$W=\frac{D\pi^4}{8}ab\sum\sum\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 C_{mn}^2.$$

If the plate is subjected to a single concentrated load P at a point $x=\xi$, $y=\eta$, the work done by the external forces is

$$P\sum\sum C_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}.$$

The force function will be expressed as

$$F=P\sum\sum C_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}-\frac{D\pi^4}{8}ab\sum\sum C_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2.$$

---

[1] Here we satisfy not only geometrical but also statical conditions on the contour, since the supporting bending moments M_1 and M_2 vanish.

By setting its derivatives with respect to the coefficients C_{mn} equal to zero, we obtain a system of equations of the form

$$P \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} - \frac{D\pi^4}{4} ab \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 C_{mn} = 0,$$

whence we find the coefficients C_{mn}:

$$C_{mn} = \frac{4Pa^3b^3}{D\pi^4 (m^2b^2 + n^2a^2)^2} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} .$$

Substituting this in (11.30), we obtain the equation of the deflected middle plane of the plate:

$$w = \frac{4Pa^3b^3}{D\pi^4} \sum \sum \frac{\sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}}{(m^2a^2 + n^2b^2)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} .$$

If we assume here that $m = 1, 2, 3, \ldots, \infty$ and $n = 1, 2, 3, \ldots, \infty$, we obtain the exact solution of Navier presented in Sec. 73 of Chapter X; retaining the first term alone in (11.30), we find the approximate equation:

$$w = \frac{4Pa^3b^3}{D\pi^4} \frac{\sin \frac{\pi\xi}{a} \sin \frac{\pi\eta}{b}}{(a^2 + b^2)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} .$$

If the load is applied at the centre $\left(\xi = \frac{a}{2},\ \eta = \frac{b}{2} \right)$, the maximum deflection will be at the same point:

$$w_{max} = f = \frac{4Pa^3b^3}{D\pi^4 (a^2 + b^2)^2} .$$

This first approximation differs from the exact value in the case of a square plate by less than 10 per cent.

The foregoing examples refer to bars and plates. In the three-dimensional problem of the theory of elasticity in the case of equilibrium of a body of arbitrary shape the general variational equation (11.15) is applied as follows.

Let us suppose that a body is acted upon by the surface tractions

$$X_v,\quad Y_v,\quad Z_v$$

and the body forces

$$X\rho,\quad Y\rho,\quad Z\rho$$

and, in addition, geometrical constraints (for instance, fixing) are imposed on it, because of which the displacements of some of its points are predetermined; for simplicity, we assume that these constraints are imposed only on the surface of the body; hence on the whole surface or part of it where the constraints

are imposed the displacements must have prescribed values

$$u = \bar{u}(x, y, z); \quad v = \bar{v}(x, y, z); \quad w = \bar{w}(x, y, z), \tag{11.31}$$

for instance, they must vanish.

Let

$$u_0(x, y, z); \quad v_0(x, y, z); \quad w_0(x, y, z) \tag{11.32}$$

now be arbitrary functions; if constraints (11.31) are imposed, these functions must conform on the surface (or on the corresponding part of it) to the constraints, i.e., there must be

$$u_0 = \bar{u}; \quad v_0 = \bar{v}; \quad w_0 = \bar{w}; \tag{11.33}$$

furthermore, let

$$f_m(x, y, z); \quad \varphi_m(x, y, z); \quad \psi_m(x, y, z) \quad (m = 1, 2, 3, \ldots)$$

also be arbitrary functions but vanishing at the points of the surface of the body, where the constraints are imposed. Let us specify the displacements of points of the body by the following formulas:

$$\left.\begin{aligned} u &= u_0 + \Sigma a_m f_m, \\ v &= v_0 + \Sigma b_m \varphi_m, \\ w &= w_0 + \Sigma c_m \psi_m; \end{aligned}\right\} \tag{11.34}$$

a_m, b_m, c_m are arbitrary constant coefficients. Varying them, one can obtain from (11.34) a set of the states of strain in the body satisfying the conditions imposed by the constraints, since at any values of the constants a_m, b_m, c_m conditions (11.31) are satisfied.

Substituting the values (11.34) in equations (III) of Chapter IV and hence in formula (3.33) of the elastic energy

$$W = \frac{1}{2}\left[\lambda\theta^2 + 2\mu\left(e_{xx}^2 + e_{yy}^2 + e_{zz}^2\right) + \mu\left(e_{xy}^2 + e_{yz}^2 + e_{zx}^2\right)\right]. \tag{11.35}$$

we observe that the integral in the right-hand member of equation (11.15), which represents the elastic energy of the whole body, appears to be a homogeneous function V of the second degree in the constants a_m, b_m, c_m:

$$\int W d\tau = V(a_m, b_m, c_m).$$

Further, for substitution in (11.15) we calculate the variations of displacements (11.34), taking into account that only the con-

stants a_m, b_m, c_m are varied:

$$\delta u = \Sigma f_m(x, y, z)\,\delta a_m,$$
$$\delta v = \Sigma \varphi_m(x, y, z)\,\delta b_m$$
$$\delta w = \Sigma \psi_m(x, y, z)\,\delta c_m.$$

Introducing these values of the variations in equation (11.15) and collecting the coefficients of the variations of the constants δa_m, δb_m, δc_m, we obtain the system of equations of the form

$$\sum\left[\int\limits_{(S)} X_v f_m\,dS + \int\limits_{(\tau)} X\rho f_m\,d\tau - \frac{\partial V}{\partial a_m}\right]\delta a_m + \\ + \sum\left[\int\limits_{(S)} Y_v \varphi_m\,dS + \int\limits_{(\tau)} Y\rho\varphi_m\,d\tau - \frac{\partial V}{\partial b_m}\right]\delta b_m + \\ + \sum\left[\int\limits_{(S)} Z_v \psi_m\,dS + \int\limits_{(\tau)} Z\rho\psi_m\,d\tau - \frac{\partial V}{\partial c_m}\right]\delta c_m = 0, \qquad (11.36)$$

where $m = 1, 2, 3, \ldots$. Inasmuch as all variations δa_m, δb_m, δc_m are arbitrary and independent of each other, equations (11.36) are possible only if the coefficients of all variations are zero; hence we arrive at the following system of equations:

$$\left.\begin{aligned} \frac{\partial V}{\partial a_m} &= \int\limits_{(S)} X_v f_m\,dS + \int\limits_{(\tau)} X\rho f_m\,d\tau, \\ \frac{\partial V}{\partial b_m} &= \int\limits_{(S)} Y_v \varphi_m\,dS + \int\limits_{(\tau)} Y\rho\varphi_m\,d\tau, \\ \frac{\partial V}{\partial c_m} &= \int\limits_{(S)} Z_v \psi_m\,dS + \int\limits_{(\tau)} Z\rho\psi_m\,d\tau. \end{aligned}\right\} \qquad (11.37)$$

The left-hand members of these equations are linear functions of the constants a_m, b_m, c_m while the right-hand members are independent of these constants; the number of equations is equal to the number of unknowns a_m, b_m, c_m.

Since V is a homogeneous function of the second degree in a_m, b_m, c_m, the coefficients of these unknowns in the left-hand members of (11.37) form a symmetric matrix, and the system of equations (11.37) is, in the terminology of structural mechanics, canonical.

Introducing the values of the coefficients a_m, b_m, c_m, obtained from (11.37), in (11.34), we find the displacements; further, according to formulas (III), we determine the strains e_{xx}, e_{yy}, $\ldots$,

e_{xy} and, at last, from the equations

$$\left.\begin{aligned} X_x &= \lambda\theta + 2\mu e_{xx}; & Y_z &= \mu e_{yz}; \\ Y_y &= \lambda\theta + 2\mu e_{yy}; & Z_x &= \mu e_{zx}; \\ Z_z &= \lambda\theta + 2\mu e_{zz}; & X_y &= \mu e_{xy} \end{aligned}\right\} \tag{11.38}$$

we find the stresses. It is to be noted that at the surface of the body these stresses will not, in general, correspond to the prescribed loads X_v, Y_v, Z_v, since in the process of solution it is not required to satisfy the statical conditions on the surface

$$\left.\begin{aligned} X_v &= X_x \cos(vx) + X_y \cos(vy) + X_z \cos(vz), \\ Y_v &= Y_x \cos(vx) + Y_y \cos(vy) + Y_z \cos(vz), \\ Z_v &= Z_x \cos(vx) + Z_y \cos(vy) + Z_z \cos(vz). \end{aligned}\right\} \tag{11.39}$$

These conditions will be satisfied approximately, and the greater the number of terms taken in expressions (11.34) and, consequently, the greater the number of constants a_m, b_m, c_m introduced in the solution, the more exactly will these conditions be satisfied. We met with this circumstance above in considering the problem of the bending of a bar and specifying the displacements in the form (11.26) where the statical conditions $M=0$ at $x=0$ and $x=l$ were not satisfied.

80. CASTIGLIANO'S VARIATIONAL EQUATION

In Chapter IV it was pointed out that three differential equations of equilibrium

$$\left.\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X\rho &= 0, \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y\rho &= 0, \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z\rho &= 0 \end{aligned}\right\} \tag{I}$$

are inadequate for determining the stresses

$$X_x,\ Y_y,\ Z_z,\ X_y,\ Y_z,\ Z_x \tag{11.40}$$

and that the problem of the theory of elasticity is therefore statically indeterminate; hence it is possible to obtain any number of systems of stresses (11.40) satisfying equations (I) and the

prescribed conditions on the surface of a body

$$\left.\begin{aligned} X_v &= X_x \cos(vx) + X_y \cos(vy) + X_z \cos(vz), \\ Y_v &= Y_x \cos(vx) + Y_y \cos(vy) + Y_z \cos(vz), \\ Z_v &= Z_x \cos(vx) + Z_y \cos(vy) + Z_z \cos(vz). \end{aligned}\right\} \tag{II}$$

Let us introduce, in addition to (11.40), another system of stresses which differs from it by infinitesimal quantities

$$\left.\begin{aligned} &X_x + \delta X_x; \quad Y_y + \delta Y_y; \quad Z_z + \delta Z_z; \\ &X_y + \delta X_y; \quad Y_z + \delta Y_z; \quad Z_x + \delta Z_x \end{aligned}\right\} \tag{11.41}$$

and also satisfies the equations of equilibrium

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(X_x + \delta X_x) + \frac{\partial}{\partial y}(X_y + \delta X_y) + \frac{\partial}{\partial z}(X_z + \delta X_z) + X\rho = 0; \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{11.42}$$

The quantities δX_x, δY_y, δZ_z, δX_y, δY_z, δZ_x will be called the variations of stresses. For generality, we assume that stresses (11.41) are due to the load which differs from the former by an infinitesimal quantity so that on the surface

$$\left.\begin{aligned} X_v + \delta X_v &= (X_x + \delta X_x) \cos(vx) + (X_y + \delta X_y) \cos(vy) + \\ &\quad + (X_z + \delta X_z) \cos(vz), \\ Y_v + \delta Y_v &= (Y_x + \delta Y_x) \cos(vx) + (Y_y + \delta Y_y) \cos(vy) + \\ &\quad + (Y_z + \delta Y_z) \cos(vz), \\ Z_v + \delta Z_v &= (Z_x + \delta Z_x) \cos(vx) + (Z_y + \delta Z_y) \cos(vy) + \\ &\quad + (Z_z + \delta Z_z) \cos(vz), \end{aligned}\right\} \tag{11.43}$$

where δX_v, δY_v, δZ_v are the variations of surface tractions. Subtracting by members equations (I) from (11.42), and equations (II) from (11.43) we obtain:

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(\delta X_x) + \frac{\partial}{\partial y}(\delta X_y) + \frac{\partial}{\partial z}(\delta X_z) = 0, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{11.44}$$

$$\left.\begin{aligned} &\delta X_v = \delta X_x \cos(vx) + \delta X_y \cos(vy) + \delta X_z \cos(vz), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{11.45}$$

Consequently, the variations of stresses themselves must satisfy the equations of equilibrium (11.44) without body forces and the surface conditions (11.45).

Let the system of stresses (11.40) actually exist in an elastic body under the action of given surface and body forces

$$X_v,\ Y_v,\ Z_v; \quad X\rho,\ Y\rho,\ Z\rho.$$

Let us see what makes it different from the whole variety of systems of stresses (11.41) close to it and how it can be determined from this variety. To this end we revert to the fundamental equality (11.10), leaving out the forces of inertia

$$\rho\frac{\partial^2 u}{\partial t^2}, \quad \rho\frac{\partial^2 v}{\partial t^2}, \quad \rho\frac{\partial^2 w}{\partial t^2}$$

since we are dealing with the equilibrium of a body.

We take for the displacements u', v', w' and the corresponding strains e'_{xx}, e'_{yy}, ..., e'_{zx} involved in (11.10) the displacements and strains actually existing in an elastic body and denote them, as usual, by

$$\left.\begin{array}{c} u,\ v,\ w; \\ e_{xx},\ e_{yy},\ e_{zz},\ e_{xy},\ e_{yz},\ e_{zx}; \end{array}\right\} \qquad (11.46)$$

we connect the strains with the actually existing stresses by Hooke's law:

$$\left.\begin{array}{ll} e_{xx}=\frac{1}{E}[X_x-\sigma(Y_y+Z_z)]; & e_{xy}=\frac{2(1+\sigma)}{E}X_y; \\ e_{yy}=\frac{1}{E}[Y_y-\sigma(Z_z+X_x)]; & e_{yz}=\frac{2(1+\sigma)}{E}Y_z; \\ e_{zz}=\frac{1}{E}[Z_z-\sigma(X_x+Y_y)]; & e_{zx}=\frac{2(1+\sigma)}{E}Z_x; \end{array}\right\} \qquad (\mathrm{V})$$

Further, we take for the stresses denoted in (11.10) by X_x, Y_y, Z_z, X_y, Y_z, Z_x the variations

$$\delta X_x,\ \delta Y_y,\ \delta Z_z,\ \rho X_y,\ \delta Y_z,\ \delta Z_x \qquad (11.47)$$

of the actual stresses appearing in the right-hand members of (V); we can do this because variations (11.47), according to the condition, satisfy the equations of equilibrium (11.44). Accordingly, X_v, Y_v, Z_v have to be replaced in (11.45) by the variations

$$\delta X_v,\ \delta Y,\ \delta Z_v$$

of the loads specified on the surface of the body. Finally, the body forces $X\rho$, $Y\rho$, $Z\rho$ must also be replaced by their variations

$$\delta(X\rho),\ \delta(Y\rho),\ \delta(Z\rho);$$

however, we have assumed above that the body forces are identical in the neighbouring states of stress [cf. equations (I) and (11.42)]; therefore, these variations should be set equal to zero; it follows that the second integral on the left-hand side of equality (11.10) vanishes and may be rewritten as

$$\int_{(S)}(\delta X_v u+\delta Y_v v+\delta Z_v w)\,dS=\int_{(\tau)}(\delta X_x e_{xx}+\delta Y_y e_{yy}+$$
$$+\delta Z_z e_{zz}+\delta X_y e_{xy}+\delta Y_z e_{yz}+\delta Z_x e_{zx})\,d\tau. \qquad (11.48)$$

We may note that the integral in the right-hand member represents the variation of the doubled elastic energy of the body

$$\frac{1}{2}\int_{(\tau)} (X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + X_y e_{xy} + Y_z e_{yz} + Z_x e_{zx})\, d\tau$$

[cf. formula (3.32) of Chapter III] on the assumption that the strains (and displacements) are not subject to variation, and the stresses alone are varied, their variations satisfying the equations of equilibrium (11.44) [1].

We now introduce the expressions of strains in terms of stresses (V) in the integrand on the right-hand side of (11.10); this integral will then be expressed as

$$\frac{1}{E}\int_{(\tau)} [X_x\delta X_x + Y_y\delta Y_y + Z_z\delta Z_z - \sigma(Y_y\delta X_x + X_x\delta Y_y + Y_y\delta Z_z + \\ + Z_z\delta Y_y + X_x\delta Z_z + Z_z\delta X_x) + 2(1+\sigma)(X_y\delta X_y + \\ Y_z\delta Y_z + Z_x\delta Z_x)]\, d\tau. \qquad (11.49)$$

But it is obvious that

$$X_x\delta X_x = \delta\left(\frac{X_x^2}{2}\right); \quad Y_y\delta Y_y = \delta\left(\frac{Y_y^2}{2}\right); \quad Z_z\delta Z_z = \delta\left(\frac{Z_z^2}{2}\right);$$

$$Y_y\delta X_x + X_x\delta Y_y = \delta(X_x Y_y), \quad \text{etc.}$$

$$X_y\delta X_y = \delta\left(\frac{X_y^2}{2}\right), \quad \text{etc.}$$

Substituting these values in integral (11.49) and putting the variation sign δ before the brackets and then before the integral sign, we represent it as

$$\delta\left\{\frac{1}{2E}\int_{(\tau)} [X_x^2 + Y_y^2 + Z_z^2 - 2\sigma(X_xY_y + Y_yZ_z + Z_zX_x) + \\ + 2(1+\sigma)(X_y^2 + Y_z^2 + Z_x^2)]\, d\tau\right\}. \qquad (11.50)$$

Comparing this with formula (3.34) of Chapter III, we see that the integrand represents the elastic energy per unit volume of the body, and integral (11.50) can be written as

$$\delta\int_{(\tau)} W\, d\tau.$$

[1] This means that in the process of variation we compare the actual state of stress with others which are close to it and also satisfy the equations of equilibrium.

Substituting this in the right-hand member of (11.48), we obtain Castigliano's variational equation

$$\int_{(S)} (\delta X_v u + \delta Y_v v + \delta Z_v w)\, dS = \delta \int_{(\tau)} W\, d\tau. \qquad (11.51)$$

The left-hand member of this equality represents the work done by the variations of the surface tractions on the actual displacements; the right-hand member involves the variation of the elastic energy of the body resulting from the variation of the actual state of stress.

The right-hand member of (11.51) may conveniently be put in the form of (11.49) where the variations of the stresses X_x, Y_y, ..., Z_x must satisfy the equations of equilibrium (11.44) while the variations of the surface tractions

$$X_v, \quad Y_v, \quad Z_v$$

must be calculated by formulas (11.45) in terms of the variations of stresses.

The variational equation (11.51) gives the desired relation between the actual stressed state of equilibrium of an elastic body and the neighbouring states which also satisfy the equilibrium conditions.

In the discussion above we did not touch upon the compatibility conditions [equations (IV) of Chapter IV], which take the form of Beltrami's equations for an isotropic body in the absence of body forces (Chapter V, Sec. 36). These equations should also be satisfied; it is important to note, however, that the compatibility equations (IV) appear to be a consequence of the variational equation (11.51) and can be deduced therefrom [1]. Consequently, when applying the variational equation (11.51) there is no need to be concerned with satisfying the conditions of compatibility (continuity of strain components).

81. APPLICATION OF CASTIGLIANO'S VARIATIONAL EQUATION TO THE PROBLEM OF TORSION OF A PRISMATICAL ROD

To apply Castigliano's principle in this problem it is necessary, first of all, to assign a system of stresses satisfying the equations of equilibrium (I) and the surface conditions; this state of stress can easily be obtained by using Prandtl's stress function $U(x, y)$

[1] See L. S. Leibenzon, *Theory of Elasticity*, 2nd ed., Moscow, Gostekhizdat, 1947, Chap. XI, Sec. 122.

[formula (8.16)]:

$$\left.\begin{aligned} &X_x = Y_y = Z_z = X_y = 0; \\ &X_z = \frac{\partial U}{\partial y}; \quad Y_z = -\frac{\partial U}{\partial x}. \end{aligned}\right\} \tag{11.52}$$

We confine ourselves to the case of a simply connected cross section; the equilibrium equations are then satisfied; should we also require the function U to vanish on the contour, then the lateral surface of the rod will be free of loads and, consequently, the conditions on it will be satisfied. The conditions at the ends of the rod, as we saw in Chapter VIII, are prescribed, according to Saint-Venant's principle, in integral form only: the resultant force vector at either of the bases is equal to zero, and the resultant moment vector is expressed in terms of the stress function:

$$M_z = 2\int\int U\,dx\,dy. \tag{11.53}$$

The displacements of the points of the rod, according to (8.1) and (8.2), are given by the formulas

$$u = -\tau yz; \quad v = \tau xz; \quad w = \tau\varphi(x, y). \tag{11.54}$$

The variation of stresses (11.52) will be obtained by giving the variation δU to the stress function; then

$$\begin{gathered} \delta X_x = \delta Y_y = \delta Z_z = \delta X_y = 0; \\ \delta X_z = \frac{\partial}{\partial y}(\delta U); \quad \delta Y_z = -\frac{\partial}{\partial x}(\delta U) \end{gathered}$$

and the variation of the elastic energy (11.50) will be expressed as

$$\delta\left\{\frac{2(1+\sigma)}{2E}\int\limits_{(\tau)}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2\right]d\tau\right\}.$$

The stresses X_v, Y_v, Z_v on the lateral surface are equal to zero and, if the variation of the stress function vanishes here, we have

$$\delta X_v = \delta Y_v = \delta Z_v = 0$$

and, consequently, the surface integral in the left-hand member of (11.48), taken over the lateral surface, becomes zero. Let us now turn to the bases of the rod; the stress distribution is identical in all cross sections, the bases of the rod included; hence, if we vary stresses (11.52), the conditions on the bases will also be varied; the integral in the left-hand member of (11.48) will be different from zero, and it should be evaluated. On the bases we have, from (II),

$$X_v = X_z; \quad Y_v = Y_z; \quad Z_v = 0.$$

Let us assume that the lower base $z=0$ is fixed in its plane; then $u=v=0$ and the corresponding integral becomes zero. It remains to evaluate the integral over the upper base $z=l$ where l is the length of the rod; therefore, putting $z=l$ in (11.54), we write down equation (11.51) as

$$\int\limits_{(S)} (\delta X_v u + \delta Y_v v + \delta Z_v w)\, dS = -l\tau \int\limits_{(S)} \left[\frac{\partial}{\partial y}(\delta U)\, y + \frac{\partial}{\partial x}(\delta U)\, x\right] dS. \tag{11.55}$$

Consider the first of the integrals on the right-hand side:

$$\int\int y \frac{\partial}{\partial y}(\delta U)\, dx\, dy = \int\limits_{x_1}^{x_2} dx \int\limits_{y_1}^{y_2} y \frac{\partial}{\partial y}(\delta U)\, dy. \tag{11.56}$$

We apply integration by parts to the inner integral:

$$\int\limits_{y_1}^{y_2} y \frac{\partial}{\partial y}(\delta U)\, dy = y\delta U \Big|_{y_1}^{y_2} - \int\limits_{y_1}^{y_2} \delta U\, dy.$$

Since, according to the condition, the variation δU is zero on the contour, the first term in this formula vanishes (Fig. 102), and we have

$$\int y \frac{\partial}{\partial y}(\delta U)\, dy = -\int\limits_{y_1}^{y_2} \delta U\, dy.$$

Substituting this in (11.56), we find that

$$\int\int y \frac{\partial}{\partial y}(\delta U)\, dy = -\int\int \delta U\, dx\, dy.$$

If we apply an analogous transformation to the second term in the right-hand member of (11.55), we obtain the same result; therefore, (11.55) yields

$$\int\limits_{(S)} (\delta X_v u + \delta Y_v v + \delta Z_v w)\, dS = -2\tau l \int\int \delta U\, dx\, dy.$$

The variational equation (11.51) takes the form

$$\delta\left\{\frac{1}{2\mu}\int\int\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2\right] dx\, dy\right\} = -2\tau l\delta \int\int U\, dx\, dy.$$

It is taken into account here that

$$\frac{2(1+\sigma)}{E} = \frac{1}{\mu}$$

and that when integrating with respect to z the stress distribution is independent of z. Dividing the preceding equation by $\frac{l}{\mu}$, we transpose all the terms to the left-hand side and put the variation sign before the braces:

$$\delta\left\{\frac{1}{2}\int\int\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2-4\mu\tau U\right]dx\,dy\right\}=0. \qquad (11.57)$$

This equality requires that the function $U(x, y)$, corresponding to the actual stress distribution in torsion, should minimise the

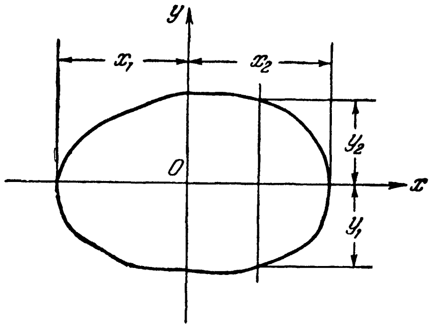

Fig. 102

value of the integral appearing in the braces. The condition (11.57) is equivalent to the differential equation (8.67)

$$\nabla^2 U=-2\mu\tau$$

with the condition on the contour $U=0$; this can be proved. In fact, carrying out variation in (11.57), we obtain

$$\int\int\left[\frac{\partial U}{\partial x}\,\delta\left(\frac{\partial U}{\partial x}\right)+\frac{\partial U}{\partial y}\,\delta\left(\frac{\partial U}{\partial y}\right)-2\mu\tau\,\delta U\right]dx\,dy=0. \quad (11.58)$$

Applying integration by parts and interchanging the order of variation and differentiation, we have

$$\int\int\frac{\partial U}{\partial x}\,\delta\left(\frac{\partial U}{\partial x}\right)dx\,dy=\int dy\int\frac{\partial U}{\partial x}\,\frac{\partial}{\partial x}(\delta U)\,dx=$$

$$=\int dy\left[\frac{\partial U}{\partial x}\,\delta U\Big|_{x_1}^{x_2}-\int\delta U\,\frac{\partial^2 U}{\partial x^2}\,dx\right].$$

But in the substitution outside the sign of the inner integral the values of x_1 and x_2 refer to the points of the contour at which,

according to the condition, $\delta U=0$ and, therefore,

$$\int\int \frac{\partial U}{\partial x}\delta\left(\frac{\partial U}{\partial x}\right)dx\,dy=-\int\int \frac{\partial^2 U}{\partial x^2}\delta U\,dx\,dy.$$

Similarly, we obtain

$$\int\int \frac{\partial U}{\partial y}\delta\left(\frac{\partial U}{\partial y}\right)dx\,dy=-\int \frac{\partial^2 U}{\partial y^2}\delta U\,dx\,dy.$$

Substituting this in (11.58) and taking δU out of the brackets, we have

$$-\int\int\left[\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+2\mu\tau\right]\delta U\,dx\,dy=0.$$

But since the variation δU is arbitrary, the integral on the left-hand side of this equality may vanish only if the bracketed expression is equal to zero at all points of the region of integration, i. e., at all points of the cross section. Hence, for all points of the section

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=-2\mu\tau;$$

this coincides with equation (8.67), which is the compatibility condition in the torsion problem; this is quite natural, for, as stated at the end of Sec. 80, Saint-Venant's compatibility conditions (IV) appear to be a consequence of Castigliano's variational equation.

The variational equation (11.57) provides a very important means for solving problems of the torsion of a rod, having a complex cross section; the solution of such problems by the usual method of integrating equation (8.67) with the boundary condition $U=0$ is obtainable in extremely rare cases; in the meantime the variational equation (11.57) makes it possible to find an approximate solution in the case of a simply connected section with sufficient accuracy.

In fact, let the contour of the cross section be given by the equation

$$F(x, y)=0^1. \tag{11.59}$$

Let us take the stress function in the following form:

$$U(x, y)=F(x, y)[C_1f_1(x, y)+C_2f_2(x, y)+\dots+C_nf_n(x, y)], \tag{11.60}$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants, $f_1(x, y), f_2(x, y), \dots$ $\dots, f_n(x, y)$ are arbitrary functions. The function U satisfies the

[1] Since the section is assumed to be simply connected, the function $F(x, y)$ must not vanish inside the section.

boundary condition of the problem because it vanishes on the contour of the section. Its variation will be obtained by varying the coefficients C_m. The variation δU also vanishes on the contour. Substituting this value of U in equation (11.57) and carrying out integration, we find that the integral in its left-hand member will be expressed as a quadratic function, i.e., a polynomial of the second degree in the constants C_m:

$$\psi(C_1, C_2, \ldots, C_n) = A_{11}C_1^2 + A_{22}C_2^2 + \ldots + A_{12}C_1C_2 + \ldots$$
$$\ldots + A_{01}C_1 + A_{02}C_2 + \ldots,$$

and to obtain an approximate solution of the problem it is necessary to find a minimum of this function, i.e., to equate to zero its partial derivatives with respect to $C_1, C_2, \ldots, C_n$. Then, as mentioned in Sec. 79, we obtain a canonical system of linear equations to find the coefficients C_m, and the function U, according to (11.60), will be determined. Using this method, L. S. Leibenzon solved a number of torsional problems for bars of various cross sections. (See his *Variational Methods for Solving Problems of the Theory of Elasticity,* Moscow, Gostekhizdat, 1943.)

82. FIRST PROBLEM OF THE THEORY OF ELASTICITY; SECOND THEOREM OF MINIMUM ENERGY

The first problem of the theory of elasticity was defined in Sec. 37 of Chapter V as the case where loads are prescribed on the surface of a body; these loads, according to the meaning of the problem, must be the same both in the actual state of the body and in all neighbouring states which are obtained by variation; hence, we must equate to zero the variations of loads X_v, Y_v, Z_v on the whole surface of the body in equation (11.51); then we get the following variational equation:

$$\delta \int_{(\tau)} W\, d\tau = 0. \tag{11.61}$$

This is a necessary condition for the elastic energy of a body in the actual state of equilibrium to have a maximum or a minimum value in comparison with all possible neighbouring states of equilibrium; this equation will be obtained in expanded form by setting integral (11.49) equal to zero. To decide whether the elastic energy is a maximum or a minimum, we form the total variation of the elastic energy. The variation given by formula (11.49) is the first variation; we see that it is formed as the first

differential of a function of several independent variables in a formal sense; for example, the first differential of a function $z(x, y)$ is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

The second differential will be obtained by differentiating dz with respect to x and y, but considering dx and dy constant:

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + \frac{\partial^2 z}{\partial x \partial y} dx\, dy + \frac{\partial^2 z}{\partial y \partial x} dy\, dx + \frac{\partial^2 z}{\partial y^2} dy^2.$$

By analogy with this we get the second variation of the energy

$$\int\limits_{(\tau)} W\, d\tau$$

by varying the stresses $X_x, Y_y, \ldots, Z_x$ in integral (11.49) but considering $\delta X_x, \delta Y_y, \ldots, \delta Z_x$ constant; consequently,

$$\delta^2 \left[\int W\, d\tau\right] = \frac{1}{E}\int [\delta X_x \delta X_x + \delta Y_y \delta Y_y + \delta Z_z \delta Z_z -$$
$$-\sigma(\delta Y_y \delta X_x + \delta X_x \delta Y_y + \ldots) + 2(1+\sigma)(\delta X_y \delta X_y + \ldots)]\, d\tau =$$
$$= \frac{1}{E}\int \{(\delta X_x)^2 + (\delta Y_y)^2 + (\delta Z_z)^2 - 2\sigma(\delta X_x \delta Y_y + \delta Y_y \delta Z_z + \delta X_x \delta Z_z) +$$
$$+ 2(1+\sigma)[(\delta X_y)^2 + (\delta Y_z)^2 + (\delta Z_x)^2]\}\, d\tau.$$

It will be noted that the expression in the braces represents the elastic energy per unit volume produced by the stresses

$$\delta X_x,\ \delta Y_y, \ \ldots\ \delta Z_x.$$

At the end of Sec. 21 of Chapter III it was pointed out that the elastic energy is always positive; hence

$$\delta^2 \left(\int\limits_{(\tau)} W\, d\tau\right) > 0. \tag{11.62}$$

Reverting to the analogy with differentiation of functions, we recall that the condition

$$dz = 0$$

is a necessary condition for a maximum or a minimum of the function z; if it turns out that

$$d^2 z > 0,$$

then the function z has a minimum. Thus conditions (11.61) and (11.62) show that the actual stressed state of equilibrium of an

elastic body differs from all neighbouring states of equilibrium in that it minimises the elastic energy. This is the principle of least work known in structural mechanics. To apply it, the elastic energy, according to (11.50), (11.51) and (11.61), should be expressed as a function of stresses:

$$W=\frac{1}{2E}\left[X_x^2+Y_y^2+Z_z^2-2\sigma(X_xY_y+Y_yZ_z+Z_zX_x)+\right.$$
$$\left.+2(1+\sigma)(X_y^2+Y_z^2+Z_x^2)\right]. \qquad (11.63)$$

The problems of structural mechanics deal with rods or bars to which, when subjected to tension, compression and bending, it is permissible to apply the hypothesis of plane sections; this makes it possible to express the elastic energy as a function of bending and twisting moments and also longitudinal and shearing forces rather than as a function of stresses:

$$W=\frac{1}{2}\left[\frac{M^2}{EJ}+\frac{N^2}{EF}+\frac{Q^2}{GF\alpha}+\frac{M_t^2}{GC}\right];$$

here C is the torsional rigidity of the bar; the energy is related to unit length of the axis of the bar and not to unit volume; therefore, the variational equation (11.61) requires in this case a minimum of the integral

$$\int\left(\frac{M^2}{EJ}+\frac{N^2}{EF}+\frac{Q^2}{GF\alpha}+\frac{M_t^2}{GC}\right)ds \qquad (11.62')$$

taken throughout the length s of the axis of the bar; this condition, as is known, permits the solution of statically indeterminate problems. The general problem of the theory of elasticity for a body on the surface of which there are given loads is also statically indeterminate; to solve it one may apply an approximate method based on the variational equation (11.61) and analogous to the Ritz-Timoshenko method presented in Sec. 79.

83. APPROXIMATE METHOD BASED ON VARIATIONAL EQUATION (11.61)

This method has been proposed by P. F. Papkovitch and consists in the following. Let us attempt to find a state of stress in a body satisfying the equations of equilibrium (I) and the conditions specified on the surface of the body; we denote the components of the corresponding stress tensor by

$$X_x^{(0)},\ Y_y^{(0)},\ Z_z^{(0)},\ Y_z^{(0)},\ Z_x^{(0)},\ X_y^{(0)}. \qquad (11.64)$$

Let us now find several states of stress with the components

$$X_x^{(m)},\ Y_y^{(m)},\ Z_z^{(m)},\ Y_z^{(m)},\ Z_x^{(m)},\ X_y^{(m)}, \tag{11.65}$$

where $m=1, 2, 3, \ldots$.

These states of stress must also satisfy the equations of equilibrium (I), but at the same time the whole surface of the body must remain free of stresses so that on the surface, according to (II),

$$\left.\begin{aligned} X_x^{(m)} \cos(vx) + X_y^{(m)} \cos(vy) + X_z^{(m)} \cos(vz) = 0, \\ Y_x^{(m)} \cos(vx) + Y_y^{(m)} \cos(vy) + Y_z^{(m)} \cos(vz) = 0, \\ Z_x^{(m)} \cos(vx) + Z_y^{(m)} \cos(vy) + Z_z^{(m)} \cos(vz) = 0, \end{aligned}\right\} \tag{11.66}$$

The states of stress (11.64) and (11.65) do not satisfy, in general, the compatibility conditions, i.e., Beltrami's equations (VII) or Mitchell's equations (Chapter V, Sec. 36) because this requirement would make the solution of the problem[1] extremely difficult, but it is not obligatory, as we have seen, in Castigliano's method; at the same time the state of stress (11.65) subject to conditions (11.66), as such, is impossible, since it would give initial stresses ("self-stressed state"), whose absence is assumed in advance (see Sec. 21) according to the hypothesis of the natural state. We shall not, however, consider this state of stress by itself but, using (11.64) and (11.65), we shall construct a new state of stress

$$\left.\begin{aligned} X_x &= X_x^{(0)} + \Sigma C_m X_x^{(m)}; & Y_z &= Y_z^{(0)} + \Sigma C_m Y_z^{(m)}; \\ Y_y &= Y_y^{(0)} + \Sigma C_m Y_y^{(m)}; & Z_x &= Z_x^{(0)} + \Sigma C_m Z_x^{(m)}; \\ Z_z &= Z_z^{(0)} + \Sigma C_m Z_z^{(m)}; & X_y &= X_y^{(0)} + \Sigma C_m X_y^{(m)}, \end{aligned}\right\} \tag{11.67}$$

where C_m are arbitrary constants.

It is evident that stresses (11.67):

(1) satisfy the equations of equilibrium;

(2) satisfy the conditions on the surface of the body;

(3) contain linearly several arbitrary constants C_m, permitting the state of stress of the body to be varied so as to satisfy the conditions of equilibrium and the surface conditions, which is required for Castigliano's variational equation (11.61) to apply, i. e., for finding a minimum of the elastic energy of the body. The greater the number of the states of stress (11.65) and of the corresponding constants C_m we introduce, the wider the possibilities of varying the energy, the closer the solution obtained to

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[1] If stresses (11.64) satisfied the compatibility conditions, they would give a complete solution of the problem.

the exact solution and the more closely we come to the satisfaction of the compatibility conditions.

If we substitute the values of stresses (11.67) in the expression of the elastic energy

$$W=\frac{1}{2E}\int\limits_{(\tau)}[X_x^2+Y_y^2+Z_z^2-2\sigma(X_xY_y+Y_yZ_z+Z_zX_x)+ \\ +2(1+\sigma)(X_y^2+Y_z^2+Z_x^2)] \quad (11.68)$$

and carry out integration, it will be represented as a function (polynomial) of the second degree in the constants C_m:

$$W=\Sigma\Sigma A_{ij}C_iC_j+\Sigma B_kC_k;$$

$$i=1,\ 2,\ \ldots,\ N;\quad j=1,\ 2,\ \ldots,\ N;\quad k=1,\ 2,\ \ldots,\ N;$$

let us write down this expression in extended form:

$$\left.\begin{array}{l} W=A_{11}C_1^2+A_{12}C_1C_2+A_{13}C_1C_3+\ldots+A_{1N}C_1C_N+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +A_{m1}C_mC_1+A_{m2}C_mC_2+\ldots+A_{mn}C_mC_n+\ldots+A_{mN}C_mC_N+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +A_{n1}C_nC_1+A_{n2}C_nC_2+\ldots+A_{nm}C_nC_m+\ldots+A_{nN}C_nC_N+ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +B_1C_1+B_2C_2+\ldots+B_mC_m+\ldots+B_nC_n+\ldots \\ \qquad\qquad\ldots+B_NC_N+E. \end{array}\right\} \quad (11.69)$$

To obtain a minimum of the elastic energy we must equate to zero the derivatives of expression (11.69) with respect to all constants C_m: this will give a system of N linear equations for the determination of these constants, such as

$$\left.\begin{array}{l} \frac{\partial W}{\partial C_m}=A_{m1}C_1+A_{m2}C_2+\ldots+A_{mn}C_n+\ldots \\ \qquad\qquad\ldots+A_{mN}C_N+B_m=0, \\ \frac{\partial W}{\partial C_n}=A_{n1}C_1+A_{n2}C_2+\ldots+A_{nm}C_m+\ldots \\ \qquad\qquad\ldots+A_{nN}C_N+B_n=0. \end{array}\right\} \quad (11.70)$$

We note that from the first equation of (11.70) we have

$$\frac{\partial^2 W}{\partial C_m\,\partial C_n}=A_{mn},$$

and from the second

$$\frac{\partial^2 W}{\partial C_n\,\partial C_m}=A_{nm},$$

therefore,

$$A_{mn}=A_{nm}. \quad (11.71)$$

Hence, the system of equations (11.70) is canonical, according to the terminology of structural mechanics; this will provide an important means of checking rather complicated calculations of the coefficients A_{ij} of system (11.70).

It should be noted that equations (11.70) may be regarded as generalisation of canonical equations of structural mechanics; this generalisation is naturally brought about as one passes from a bar system to a body of arbitrary shape; the constants C_m play the role of redundant unknowns, but their number is infiinitely large in general, since an elastic body is a system with an infiinite number of redundant unknowns. By assigning a finite number of unknowns C_m in sums (11.67), we come to an approximate solution of the problem by reducing the number of redundant unknowns. Formulas (11.67) themselves are a generalisation of the formulas for moments of transverse and longitudinal forces in bar systems; if, for instance, the redundant unknowns of such a system are denoted by C_1, C_2, C_3, $\ldots$, C_N, the moment at a given section of the system will be expressed by a formula similar to (11.67):

$$M = M^{(0)} + C_1 M_1 + C_2 M_2 + \ldots + C_N M_N = M^{(0)} + \Sigma C_m M_m, \quad (11.72)$$

where $M^{(0)}$ is the moment of a given load; M_1, M_2, $\ldots$, M_N are the moments due to unit loads $C_1 = 1$, $C_2 = 1$, $\ldots$, $C_N = 1$.

The coefficients A_{ij} in (11.69) and (11.70), which are integrals of squares and pairwise products of functions involved in expressions (11.67), represent a generalisation of known Mohr's integrals

$$\int \frac{MM'}{EJ}\, ds; \quad \int \frac{NN'}{EF}\, ds; \quad \int \frac{QQ'}{GF}\, ds.$$

The analogy between (11.67) and (11.72) shows that the state of stress (11.64), which satisfies the equations of equilibrium and the boundary conditions but does not satisfy the compatibility conditions, corresponds, in the case of the bar system, to the state of stress in the basic statically determinate system of the force method (when the redundant constraints are removed in the given system). The state of stress (11.65) is analogous to the "unit" states of stress of the basic system which are due to the action of unit loads $C_m = 1$.

Consequently, these quantities may be interpreted as generalised forces in formulas (11.67).

All the arguments of this section rest upon the condition that the stresses on the surface of a body are not varied since it is assumed that they are given. However, in the case where the semi-inverse method is applied the stress distribution sometimes is not given on certain parts of the surface, only the resultant force vector (or the resultant) and the resultant moment vector being given on these parts of the surface. For instance, in Chapter VIII, where we considered the problems of torsion and bending of a prismatical bar, we prescribed on its bases: in bending, the load Q subject to the condition that the moment of the tangential tractions producing it be equal to zero; in torsion, the twisting moment M_z subject to the condition that the resultant force vector of the tangential tractions producing it be equal to

zero. The stress distribution in all cross sections of the bar appears to be identical; thus, varying the stresses throughout the bar, we must admit their variation on its bases as well. In such cases it becomes necessary to resort to the variational equation of the general form (11.51) instead of (11.61). The next section is concerned with the application of Castigliano's method to the general problem of a bar of rectangular cross section.

84. LAMÉ'S PROBLEM FOR AN ELASTIC RECTANGULAR PRISM

Imagine an elastic rectangular prism whose surface is acted upon by loads either normal or tangent to its faces; it is required to determine the stresses at each point of the prism. Lamé formulated this problem in 1852 and emphasised its great importance in the sense of practical application, since its solution would provide a means of checking the approximate elementary methods of investigating the bending and torsion of beams of rectangular cross section and also the compression of columns by a nonuniform load. At the same time, Lamé pointed out the extreme difficulties this problem involved in the solution of the differential equations (VI) with the conditions specified on the surface of the prism; indeed, the problem thus formulated has yet to be solved. In the meantime Castigliano's method permits the solution of Lamé's problem to be carried through in some particular cases of loadings on the faces of the prism. Such a solution, as ascertained above, will be approximate because in practice we have to work with a finite number of terms in the sums of formulas (11.67) and, consequently, with a finite number of coefficients C_m that are varied. The larger this number, the better the compatibility conditions will be satisfied; at the same time, there are great possibilities here for satisfying the boundary conditions on the faces of the prism, and this is important, if all three basic dimensions of the prism are of the same order and the alleviation of the boundary conditions by the use of Saint-Venant's principle is not justified. Let us consider the procedure of solving the problem in greater detail.

According to (11.67), the stress tensor should be represented as the sum of two tensors:

$$\begin{pmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{pmatrix} = \begin{pmatrix} X_x^{(0)} & Y_x^{(0)} & Z_x^{(0)} \\ X_y^{(0)} & Y_y^{(0)} & Z_y^{(0)} \\ X_z^{(0)} & Y_z^{(0)} & Z_z^{(0)} \end{pmatrix} + \Sigma C_m \begin{pmatrix} X_x^{(m)} & Y_x^{(m)} & Z_x^{(m)} \\ X_y^{(m)} & Y_y^{(m)} & Z_y^{(m)} \\ X_z^{(m)} & Y_z^{(m)} & Z_z^{(m)} \end{pmatrix}. \quad (11.73)$$

The first, which satisfies the equations of equilibrium (I) and the given boundary conditions, will briefly be termed the *basic tensor*. Its construction offers no difficulty in principle but it is found to be more or less involved depending on the complexity of given loads. The second tensor on the right-hand side of (11.73) must also satisfy the equations of equilibrium (I); it does not depend on the given load, since the surface of the prism must be free of stresses; hence it can be constructed once and for all for a given prism: we shall call it the *correcting tensor*. Consider one of the methods of constructing it [1].

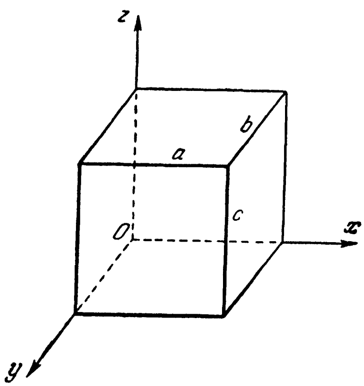

Fig. 103

Let a prism be given with the edges of length a, b, c (Fig. 103). Let us set up the following three systems of functions:

$$\left.\begin{aligned} P_m(x) &= \cos\frac{m\pi x}{a} - \cos\frac{(m+2)\pi x}{a} \quad (m = 0, 1, 2, \ldots), \\ P_n(y) &= \cos\frac{n\pi y}{b} - \cos\frac{(n+2)\pi y}{b} \quad (n = 0, 1, 2, \ldots), \\ P_p(z) &= \cos\frac{p\pi z}{c} - \cos\frac{(p+2)\pi z}{c} \quad (p = 0, 1, 2, \ldots). \end{aligned}\right\} \quad (11.74)$$

They possess the following properties [2].

1. Each of these functions and its first derivative vanish at both ends of the corresponding interval a, b, c; this is also seen in the graphs of several of the first functions of systems (11.74), represented in Fig. 104.

[1] See the author's paper, Prikl. matem. i mekhan., 1951, Vol. 15, No. 2, pp. 137-148.

[2] See the author's paper, Prikl. matem. i mekhan., 1946, Vol. 10, No. 1, pp. 193-208.

2. Any function, which can be expanded in a trigonometric series (Fourier series), can be represented in the corresponding interval with the same accuracy by a polynomial composed of functions (11.74); if, for example, a function $f(x)$ is defined in the interval $x=0$, $x=a$, it is possible to choose the coefficients c_m in the equality

$$f(x)=c_0P_0(x)+c_1P_1(x)+c_2P_2(x)+\ \dots\ +c_mP_m(x)+\ \dots$$

so as to make the right-hand member represent the function $f(x)$ as exactly as one pleases. Such a property of a system of functions

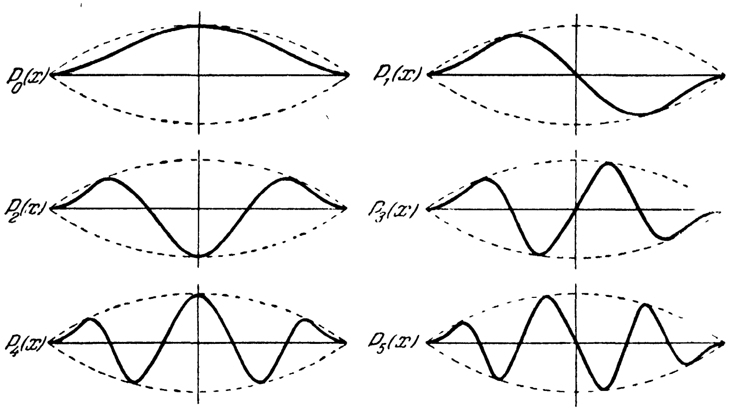

Fig. 104

is called its completeness or closeness (we shall not dwell on a certain distinction between these concepts). The system of functions (11.74) will be used to construct the correcting tensor as follows. Since it must satisfy the differential equations of equilibrium (I), we form its components with the aid of stress functions; let us make use, for instance, of Maxwell's functions φ_1, φ_2, φ_3 and write down the expressions of the components in the form of (9.4) of Sec. 56. The stress functions themselves will be represented as triple sums composed of functions (11.74):

$$\left.\begin{aligned} \varphi_1 &= \sum_m \sum_n \sum_p A_{mnp} P_m(x)\, P_n(y)\, P_p(z), \\ \varphi_2 &= \sum_m \sum_n \sum_p B_{mnp} P_m(x)\, P_n(y)\, P_p(z), \\ \varphi_3 &= \sum_m \sum_n \sum_p C_{mnp} P_m(x)\, P_n(y)\, P_p(z), \end{aligned}\right\} \qquad (11.75)$$

where A_{mnp}, B_{mnp}, C_{mnp} are arbitrary constant coefficients admitting the variation of functions (11.75). Hence, the components of the correcting tensor will be expressed as

$$\left.\begin{aligned}
X_x &= \sum_m \sum_n \sum_p [C_{mnp} P_m(x) P_n''(y) P_p(z) + \\
&\qquad + B_{mnp} P_m(x) P_n(y) P_p''(z)], \\
Y_y &= \sum_m \sum_n \sum_p [A_{mnp} P_m(x) P_n(y) P_p''(z) + \\
&\qquad + C_{mnp} P_m''(x) P_n(y) P_p(z)], \\
Z_z &= \sum_m \sum_n \sum_p [B_{mnp} P_m''(x) P_n(y) P_p(z) + \\
&\qquad + A_{mnp} P_m(x) P_n''(y) P_p(z)], \\
X_y &= -\sum_m \sum_n \sum_p C_{mnp} P_m'(x) P_n'(y) P_p(z), \\
Y_z &= -\sum_m \sum_n \sum_p A_{mnp} P_m(x) P_n'(y) P_p'(z), \\
Z_x &= -\sum_m \sum_n \sum_p B_{mnp} P_m'(x) P_n(y) P_p'(z).
\end{aligned}\right\} \qquad (11.76)$$

It can easily be proved that in the state of stress described by formulas (11.76) all the faces of the prism remain free of loads. Consider, for instance, the faces $x=0$ and $x=a$ normal to the axis Ox; they are acted upon by the stresses X_x, $Y_x=X_y$, $Z_x=X_z$ which vanish here; this follows from the properties of functions (11.76):

$$\left.\begin{aligned}
P_m(0) = 0; \quad P_m(a) = 0; \\
P_m'(0) = 0; \quad P_m'(a) = 0.
\end{aligned}\right\} \qquad (11.77)$$

In the same way we prove that the remaining two pairs of faces

$$y=0; \qquad y=b;$$
$$z=0; \qquad z=c$$

are also free of stresses at arbitrary values of the coefficients A_{mnp}, B_{mnp}, C_{mnp}. The above-mentioned completeness of the systems of functions (11.74) permits the construction, through the use of Maxwell's functions (11.75), of a very wide class of the states of stress (11.76) which leave the whole surface of the prism free of loads; thus, formulas (11.76) represent the correcting tensor in a very general form. By adding to it the basic tensor corresponding to the loads prescribed on the surface of the prism, we obtain the general tensor which satisfies the differential equations of

equilibrium of the prism, the boundary conditions of the problem formulated here and contains an unlimited number of the parameters A_{mnp}, B_{mnp}, C_{mnp} to be varied; these parameters can be determined by using Castigliano's variational method considered in Sec. 83, so that the problem is eventually reduced to the solution of a system of linear equations of the form (11.70).

It may be noted that the correcting tensor (11.76) is slightly different in form from that which was first proposed by P. F. Papkovitch in formulas (11.67); by assigning a number of numerical values 0, 1, 2, 3, ..., $N-1$ to the index m in these formulas, we introduce a simple series of the coefficients

$$C_0, C_1, C_2, \ldots, C_{N-1} \tag{11.78}$$

and for their determination we obtain a system of N linear equations in N unknowns. If, however, we assign the same numerical values to the indices m, n, p in formulas (11.76), the coefficients form a triple series and their number will be $3N^3$; the number of equations for their determination will be the same. Let us assign, for instance, *to a first approximation*, a single value, $m=n=p=0$, to all the indices; we then introduce three coefficients:

$$A_{000}, \quad B_{000}, \quad C_{000}. \tag{11.79}$$

To *a second approximation* we introduce two values of the indices: $m=0, 1$; $n=0, 1$; $p=0, 1$; then we obtain eight coefficients:

$$\left.\begin{matrix} A_{000}, & A_{100}, & A_{010}, & A_{001}; \\ A_{011}, & A_{101}, & A_{110}, & A_{111}. \end{matrix}\right\} \tag{11.80}$$

Altogether we shall introduce 24 coefficients A_{mnp}, B_{mnp}, C_{mnp}. Thus, a tendency to improve the accuracy of the solution entails a considerable increase in the number of unknowns introduced in the problem and creates difficulties of a technical character associated with the solution of systems with a large number of equations; this is due, of course, to a complicated three-dimensional nature of the problem itself. The form of solution (11.67) proposed by P. F. Papkovitch is simpler with regard to construction, but to obtain the same degree of accuracy it will, obviously, require the introduction of a considerably greater number of terms in series (11.78). The solution in the form (11.76) can be somewhat simplified by using *incomplete approximations*; for instance, in introducing a new value of the indices m, n, p we may retain only those coefficients in the triple indices of which this new value is met just once; thus, in introducing the second approximation (11.80), we retain only the coefficients of the first line where the index "1" is met just once. Then the total num-

ber of the coefficients will be $3\times4=12$, i.e., it will be reduced to one half as compared to the complete second approximation.

The application of Castigliano's method to Lamé's problem will be illustrated by the simplest example when the prism is compressed or stretched by equal loads distributed arbitrarily over two opposite faces, say, over the faces $z=0$ and $z=c$; then the boundary conditions on these faces will be

$$X_z=Y_z=0;\quad Z_z=F(x,\ y)\quad \text{for } z=c$$

$$X_{-z}=Y_{-z}=0;\quad Z_{-z}=-F(x,\ y)\quad \text{for } z=0;$$

the remaining faces must be free of loads. All these conditions will be fulfilled if the basic tensor is taken in the form

$$X_x=Y_y=X_y=Y_z=Z_x=0;\quad Z_z=F(x,\ y).$$

Substitution in the equations of equilibrium (I) shows that they are also satisfied identically. (Such a state of stress can exist if the prism is replaced by a block of the same form made up of separate bars parallel to the axis Oz on the assumption that there is neither friction nor mutual pressure between them; each bar will then carry that part of the load which acts on its ends and the stress Z_z developed in it will be constant along the length and will be independent of z; the compatibility conditions will not, of course, be satisfied then.) It remains to construct the general stress tensor as the sum of the basic (11.73) and correcting (11.76) tensors; it differs from the tensor (11.76) only in that the term $F(x,\ y)$ will be added in the formula of the component Z_z.

Fig. 105 shows a special case of a prism with a square base $a=b=1$, $c=2$ subjected to loads which are distributed over the upper and lower bases according to the law

$$F(x,\ y)=\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right).$$

The surface characterising this distribution is shown on the upper base of the prism. The load (as the prism itself) has two axes of symmetry; it is, therefore, necessary to retain in formulas (11.75) only even-numbered functions P_m, P_n, P_p having the graphs (Fig. 104) symmetrical about the middle of the segments a, b, c. The solution of this problem to a second incomplete approximation has led to a system of 12 equations[1] with respect to the co-

[1] See the author's paper, Prikl. matem. i mekhan., 1951, Vol. 15, No. 2, pp. 137-148.

efficients

$$A_{000},\quad A_{002},\quad A_{020},\quad A_{200};$$
$$B_{000},\quad B_{002},\quad B_{020},\quad B_{200};$$
$$C_{000},\quad C_{002},\quad C_{020},\quad C_{200}.$$

Fig. 105 shows the surface of distribution of stresses Z_z at the section passing through the middle of the altitude of the prism. In spite of a sharp nonuniformity of the load acting on the bases, the distribution of stresses at the midsection is sufficiently close to uniform distribution, in full accord with Saint-Venant's principle.

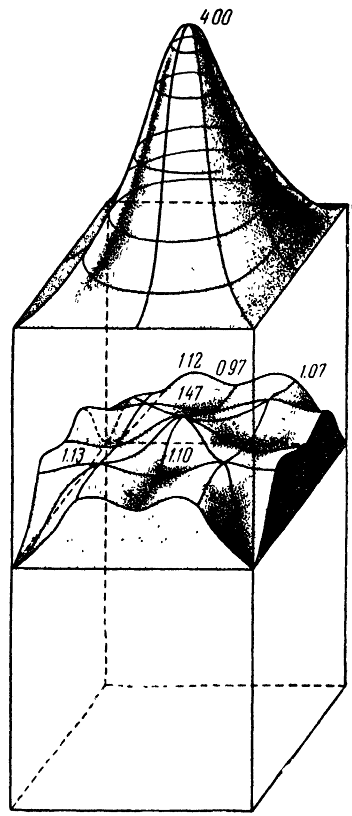

Fig. 105

Castigliano's variational method made it possible to obtain the solution of Lamé's problem for a prism in other more complicated cases of loading. V. P. Netrebko[1] treated the problems of the torsion of a rectangular prism under a given distribution of shearing stresses over its bases, and also the cases of the so-called torsion with warping restraint when one or both bases cannot warp (as follows from Saint-Venant's theory) but must remain plane E. S. Kononenko[2] found the solution of the problem of a prism compressed between two absolutely rigid plates in the presence of complete cohesion on the surfaces of contact; the problem was solved to a second complete approximation (with 24 coefficients) and the stress distribution within the prism was first clarified in detail in these complex loading conditions.

By using the cylindrical co-ordinates the method presented in this section can be extended to cover the cases of a circular cylinder, a tube or a tube sector subjected to given loads on the sur-

[1] Vestnik Mosk. Univ., 1954, No. 12, pp. 15-26; 1956, No. 6, pp. 11-25.
[2] In *Investigations on the Theory of Structures*, Moscow, Gosstroyizdat, 1954, No. 6, pp. 455-468; 1957, No. 7, pp. 437-466.

face[1]. The construction of the correcting tensor for such boddies can be carried out in a very general form through the use, for instance, of formulas (9.18) which contain three stress functions. Several problems of this kind were solved in the papers of V. N. Ionov, V. M. Ljubimov and E. R. Miroshnichenko[2].

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[1] The author's paper. Prikl. matem. i mekhan., 1953, Vol. 17, No. 4, pp. 465-469.

[2] V. N. Ionov, in *Investigations on the Theory of Structures*, Moscow, Gosstrojizdat, 1957, No. 7, pp. 413-436; E. R. Miroshnichenko, *The Problem of a Cylinder Compressed Between Rigid Plates Without Sliding*, Moscow, Mosk. lesotekhn. inst., 1957; V. M., Ljubimov. Prikl. matem. i mekhan., 1957, Vol. 21, No. 4.

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